Srednicki Chapter 32

QFT Problems & Solutions

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December 19, 2012

Srednicki 32.1. Consider the Noether current j^μ for the U(1) symmetry of equation 32.1 and the corresponding charge Q.

(a) Show that $e^{-i\alpha Q}\phi e^{i\alpha Q} = e^{i\alpha}\phi$.

We use the Hadamard Lemma:

$$e^{-i\alpha Q}\phi e^{i\alpha Q} = \phi + [-i\alpha Q, \phi] + \frac{1}{2!}[i\alpha Q, [i\alpha Q, \phi]] - \frac{1}{3!}[i\alpha Q, [i\alpha Q, [i\alpha Q, \phi]]] + \dots$$

This is:

$$e^{-i\alpha Q}\phi e^{i\alpha Q} = \phi + (-i\alpha)[Q, \phi] + \frac{(-i\alpha)^2}{2!}[Q, [Q, \phi]] + \frac{(-i\alpha)^3}{3!}[Q, [Q, [Q, \phi]]] + \dots \tag{32.1.1}$$

To proceed, we need to evaluate $[Q, \phi]$. We use equation 22.16:

$$[Q(x), \phi(x')] = \int d^3x \left[i\phi(x)\partial^0\phi^\dagger(x) - i\phi^\dagger(x)\partial^0\phi(x), \phi(x')\right]$$

Recall that $\partial^0\phi = -\dot{\phi}$. Then,

$$[Q, \phi] = \int d^3x \left[-i\phi\dot{\phi}^\dagger + i\phi^\dagger\dot{\phi}, \phi\right]$$

Next recall that $\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$. This gives $\Pi = \dot{\phi}^\dagger$ and $\Pi^\dagger = \dot{\phi}$. Then,

$$[Q, \phi] = i\int d^3x \left[-\phi\Pi + \phi^\dagger\Pi^\dagger, \phi\right]$$

This second term vanishes, since ϕ and $\phi^\dagger$ act as separate fields. Thus,

$$[Q, \phi] = -i\int d^3x \left[\phi\Pi, \phi\right]$$

For the same reason, this reduces to:

$$[Q, \phi] = -i\int d^3x\phi \left[\Pi, \phi\right]$$

Now use equation 3.28:

$$[Q,\phi] = i\int d^3x\phi i\delta^3(x-x')$$

Hence:

$$[Q,\phi] = -\phi$$

Using this in equation (32.1.1), we have:

$$e^{-i\alpha Q}\phi e^{i\alpha Q} = \phi + (-i\alpha)(-1)\phi + \frac{(i\alpha)^2}{2!}(-1)^2\phi + \frac{(-i\alpha)^3}{3!}(-1)^3\phi + \ldots$$

This is:

$$e^{-i\alpha Q}\phi e^{i\alpha Q} = \phi + (i\alpha)\phi + \frac{(i\alpha)^2}{2!}\phi + \frac{(i\alpha)^3}{3!}\phi + \ldots$$

which is:

$$e^{-i\alpha Q}\phi e^{i\alpha Q} = e^{i\alpha}$$

as expected.

(b) Use equation 32.5 to show that $e^{-i\alpha Q}|\theta\rangle = |\theta+\alpha\rangle$.

Equation 32.5 gives us:

$$\langle\theta|\phi|\theta\rangle = \frac{1}{\sqrt{2}}ve^{-i\theta}$$

But $\theta+\alpha$ is also a vacuum, so it follows that:

$$\langle\theta+\alpha|\phi|\theta+\alpha\rangle = \frac{1}{\sqrt{2}}ve^{-i(\theta+\alpha)} \tag{32.1.2}$$

Now we use the result of part (a) into equation 32.5:

$$\langle\theta|e^{i\alpha}e^{i\alpha Q}\phi e^{-i\alpha Q}|\theta\rangle = \frac{1}{\sqrt{2}}ve^{-i\theta}$$

which gives:

$$\langle\theta|e^{i\alpha Q}\phi e^{-i\alpha Q}|\theta\rangle = \frac{1}{\sqrt{2}}ve^{-i(\theta+\alpha)}$$

Using equation (32.1.2):

$$\langle\theta|e^{i\alpha Q}\phi e^{-i\alpha Q}|\theta\rangle = \langle\theta+\alpha|\phi|\theta+\alpha\rangle$$

From which it follows that:

$$e^{-i\alpha Q}|\theta\rangle = |\theta+\alpha\rangle$$

(c) Show that $Q|0\rangle \neq 0$; that is, the charge does not annihilate the $\theta = 0$ vacuum. Contrast this with the case of an unbroken symmetry.

Expanding the result from part (b) for the case that $\theta = 0$, we have:

$$|0\rangle - i\alpha Q|0\rangle + \frac{(i\alpha Q)^2}{2!}|0\rangle + \ldots = |\alpha\rangle$$

If $Q|0\rangle$ were equal to 0, then it would follow that $|0\rangle = |\alpha\rangle$, ie there is a unique ground state. This is fine for unbroken symmetries, but problematic for broken ones.

Srednicki 32.2. In problem 24.3, we showed that $[\phi_i, Q^a] = (T^a)_{ij}\phi_j$, where Q^a is the Noether charge in the SO(N) symmetric theory. Use this result to show that $Q^a|0\rangle \neq 0$ if and only if Q^a is broken.

We begin with equation 32.5, which is generally true (our renormalization condition species that v = 0 for an unbroken symmetry). Thus:

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}}v$$

Multiplying by the matrix T:

$$\langle 0|T_{ij}\phi_j|0\rangle = \frac{1}{\sqrt{2}}T_{ij}v_j$$

Now use the result from problem 24.3:

$$\langle 0|[\phi_i, Q]|0\rangle = \frac{1}{\sqrt{2}}T_{ij}v_j$$

which is:

$$\langle 0|\phi Q|0\rangle - \langle 0|Q\phi|0\rangle = \frac{1}{\sqrt{2}}T_{ij}v_j$$

Now we consider the two cases.

Case I: Broken Symmetry. In this case the right-hand side is nonzero. Hence $Q|0\rangle \neq 0$, as the alternative would destroy the equality.

Case II: Unbroken Symmetry. In this case $v = 0$ and so the right-hand side is zero. $Q|0\rangle = 0$ would certainly satisfy the equality – but is it the only possibility? Yes. ϕ is decomposed into $a^\dagger$ and a, which will leave terms of the form $\langle 0|Q|1\rangle$, which is not necessarily zero. ϕ and Q do not, in general, commute, so it is impossible to hope for a cancellation between the two terms. The only remaining option is to require that $Q|0\rangle = 0$.

Srednicki 32.3. We define the decay constant f of the Goldstone boson via

$$\langle k|j^\mu(x)|0\rangle = ifk^\mu e^{-ikx}$$

where $|k\rangle$ is the state of a single Goldstone boson with four-momentum k, normalized in the usual way, $|0\rangle$ is the $\theta = 0$ vacuum, and $j^\mu(x)$ is the Noether current.

(a) Compute f at tree level. That is, express j^μ in terms of the ρ and χ fields, then use free field theory to compute the matrix element. A nonvanishing value of f indicates that the corresponding current is spontaneously broken.

Note that we've only so far defined Goldstone bosons for the theory presented. Though the problem is ambiguous, it is reasonable to specialize to the case considered in the chapter

Equation 22.14 gives:

$$j^\mu = i\phi\partial^\mu\phi^\dagger - i\phi^\dagger\partial^\mu\phi$$

Using equation 32.8, this becomes:

$$j^\mu = -v(1+\rho/v)^2\partial^\mu\chi$$

Therefore, we have:

$$\langle k|j^\mu|0\rangle = -v(1+\rho/v)^2\partial^\mu\langle k|\chi|0\rangle$$

Using the renormalization condition decided on in chapter 5, we have:

$$\langle k|j^\mu|0\rangle = -v(1+\rho/v)^2\partial^\mu e^{-ikx}$$

Note that $v \propto 1/\sqrt{\lambda}$; in free field theory, v is divergent, so the ρ/v vanishes. Thus:

$$\langle k|j^\mu|0\rangle = -v\partial^\mu e^{-ikx}$$

It follows that f = v.

(b) Discuss how your result would be modified by higher-order corrections.

To compute the corrections, we can no longer assume that $\lambda \approx 0$. Instead, we will have to (perturbatively) compute the exact correlation function $\langle k|j\partial^\mu\chi|0\rangle$. A good place to start would be with the analog of equation 13.8.