Srednicki Chapter 3

QFT Problems & Solutions

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Srednicki 3.1. Derive the creation/annihilation commutation relations from the canonical commutation relations, the expansion of the operators, and eq. 3.24.

Use equation 3.21:

$$[a(k), a(k')] = \int d^3x d^3x' e^{-ikx} e^{-ik'x'} [i\partial_0 \phi(x) + \omega\phi(x), i\partial_0\phi(x') + \omega\phi(x')]$$

Now use equation 3.24:

$$[a(k), a(k')] = \int d^3x d^3x' e^{-ikx} e^{-ik'x'} [i\Pi(x) + \omega\phi(x), i\Pi(x') + \omega\phi(x')]$$

Now use equation 3.28, obvious that only the cross-terms survive:

$$[a(k), a(k')] = \int d^3x d^3x' e^{-ikx} e^{-ik'x'} \left([i\Pi(x), \omega\phi(x')] + [\omega\phi(x), i\Pi(x')]\right)$$

which gives:

$$[a(k), a(k')] = \int d^3x d^3x' e^{-ikx} e^{-ik'x'} i\omega \left(-[\phi(x'), \Pi(x)] + [\phi(x), \Pi(x')]\right)$$

Using 3.28 again:

$$[a(k), a(k')] = \int d^3x d^3x' e^{-ikx} e^{-ik'x'} i\omega \left(-(2\pi)^3 2\omega\delta^3(x - x') + (2\pi)^3 2\omega\delta^3(x - x')\right)$$

which is:

$$[a(k), a(k')] = 0$$

as expected. □

Now we just take the hermitian conjugate of both sides. The left side follows this formula:

$$[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[A^\dagger, B^\dagger]$$

The right side is of course unchanged, leaving:

$$[a^\dagger(k), a^\dagger(k')] = 0$$

as expected. ☐

As for the third commutator:

$$[a(k), a^\dagger(k')] = \int d^3x d^3x' e^{-ikx} e^{ik'x'} [i\Pi(x) + \omega\phi(x), -i\Pi(x') + \omega\phi(x')]$$

since ϕ is real. This gives:

$$[a(k), a^\dagger(k')] = \int d^3x d^3x' e^{-ikx} e^{ik'x'} i\omega \left([\Pi(x), \phi(x')] + [\Pi(x'), \phi(x)]\right)$$

which implies:

$$[a(k), a^\dagger(k')] = -2i \int d^3x d^3x' e^{-ikx} e^{ik'x'} i\omega\delta^3(x - x')$$

$$[a(k), a^\dagger(k')] = 2 \int d^3x d^3x' e^{ikx} e^{-ik'x'} \omega\delta^3(x - x')$$

$$[a(k), a^\dagger(k')] = 2 \int d^3x d^3x' e^{i(-\omega+\omega')t+ik\cdot x-ik'\cdot x'} \omega\delta^3(x - x')$$

$$[a(k), a^\dagger(k')] = 2 \int d^3x d^3x' e^{i(-\omega+\omega')t+ik\cdot x-ik'\cdot x'} \omega\delta^3(x - x')$$

$$[a(k), a^\dagger(k')] = 2\omega \int d^3x e^{i(-\omega+\omega')t} e^{i(k-k')\cdot x}$$

Using 3.27:

$$[a(k), a^\dagger(k')] = 2\omega(2\pi)^3\delta^3(k - k')e^{i(-\omega+\omega')t}$$

Remember that k^2 is a constant. The delta function requires that the spatial parts are equal, so the time parts (ω and ω') must be equal too. Then,

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega\delta^3(k - k')$$

as expected. ☐

Srednicki 3.2. Use the commutation relations to show explicitly that $|\phi\rangle$ is an eigenstate of the Hamiltonian, and calculate the eigenvalue. Take $\Omega_0 = \mathcal{E}_0$.

The Hamiltonian is:

$$\mathcal{H}|\phi\rangle = \int \widetilde{dk} \omega a^\dagger(k) a(k) a^\dagger(k_1) \dots a^\dagger(k_n)|0\rangle$$

Now $a(k)$ will commute with all the creation operators, leaving a "residue" of $(2\pi)^3 2\omega\delta^3(k - k_i)$ for each. When it commutes with the last one, it kills the vacuum, so only the residues are left. We have then:

$$\mathcal{H}|\phi\rangle = (2\pi)^3 2\omega \sum_{i=1}^{n} \int \widetilde{dk} \omega a^\dagger(k) a^\dagger(k_1) \dots a^\dagger(k_{i-1})\delta(k - k_i) a^\dagger(k_{i+1}) \dots a^\dagger(k_n)|0\rangle$$

Rewriting this:

$$\mathcal{H}|\phi\rangle = \sum_{i=1}^{n} \int d^3k\delta(k-k_i)\omega a^\dagger(k_1)\dots a^\dagger(k_{i-1})a^\dagger(k)a^\dagger(k_{i+1})\dots a^\dagger(k_n)|0\rangle$$

Taking the integral:

$$\mathcal{H}|\phi\rangle = \sum_{i=1}^{n} \omega_i a^\dagger(k_1)\dots a^\dagger(k_{i-1})a^\dagger(k_i)a^\dagger(k_{i+1})\dots a^\dagger(k_n)|0\rangle$$

$$\mathcal{H}|\phi\rangle = \sum_{i=1}^{n} \omega_i|\phi\rangle$$

So it is an eigenstate, with an eigenvalue of $\sum_{i=1}^{n}\omega_i$.

Problem 3.3 (based on Srednicki 3.3). The goal of this problem is to prove:

$$\mathbf{U(\Lambda)^{-1}a(k)U(\Lambda) = a(\Lambda^{-1}k)}$$
$$\mathbf{U(\Lambda)^{-1}a^\dagger(k)U(\Lambda) = a^\dagger(\Lambda^{-1}k)}$$
$$\mathbf{U(\Lambda)|k_1\dots k_n\rangle = |\Lambda k_1\dots \Lambda k_n\rangle}$$

Note: Srednicki provides little "scaffolding" for this complicated problem, yet his solution is brilliant. I have inserted the scaffolding here. If you must solve the problem in its original form, you may wish to start with equation 2.26, insert the mode expansion, then change the integration variable on the right, $k \to \Lambda k$ (this won't affect the integral since Λ is orthochronous). The first claim can then be "proved" by matching up the terms, and it is easy to prove the remaining claims given the first (see below). However, this argument is non-rigorous in its last step, which is why I present the below argument instead.

(a) Use the four-dimensional Fourier Transformation (defined in the chapter summary) and eqn. 2.26 to show $\mathbf{U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \overline{\phi}(\Lambda^{-1}k)}$.

Using the four-dimensional Fourier Transform:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = U(\Lambda)^{-1}\int d^4x e^{-ikx}\phi(x)U(\Lambda)$$

d^4k is invariant under Lorentz Transformations, so are the constants, and so is the product of two four-vectors. Hence:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \int d^4x e^{-ikx}U(\Lambda)^{-1}\phi(x)U(\Lambda)$$

Now we can use equation 2.26:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \int d^4x e^{-ikx}\phi(\Lambda^{-1}x)$$

Now define $y = \Lambda^{-1}x$.

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \int d^4x e^{-ik\Lambda y}\phi(y)$$

In the exponent, we now have $k\Lambda y = k^\mu \Lambda_\mu^{\ \nu} y_\nu$. Let's now use equation 2.5: this gives $k^\mu (\Lambda^{-1})^\nu_{\ \mu} y_\nu = y\Lambda^{-1}k$. This gives:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \int d^4x e^{-iy\Lambda^{-1}k}\phi(y)$$

Since the determinant of Λ is required by propriety to be one, we have:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \int d^4y e^{-iy\Lambda^{-1}k}\phi(y)$$

Fourier transforming the right-hand side, we obtain:

$$U(\Lambda)^{-1}\overline{\phi}(k)U(\Lambda) = \overline{\phi}(\Lambda^{-1}k)$$

(b) Verify that $\overline{\phi}(\mathbf{k}) = 2\pi\delta(\mathbf{k}^2 + \mathbf{m}^2)\left[\theta(\mathbf{k}^0)\mathbf{a}(\mathbf{k}) + \theta(-\mathbf{k}^0)\mathbf{a}^\dagger(-\mathbf{k})\right]$

The four dimensional Fourier Transformation is:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\overline{\phi}(k)$$

Plugging in the claim:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} 2\pi\delta(k^2 + m^2)\left[\theta(k^0)a(\vec{k}) + \theta(-k^0)a^\dagger(-\vec{k})\right]$$

which is:

$$\phi(x) = \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3} \int d\omega \delta(-\omega^2 + \vec{k}^2 + m^2)\left[e^{-i\omega t}\theta(k^0)a(\vec{k}) + e^{-i\omega t}\theta(-k^0)a^\dagger(-\vec{k})\right]$$

Let's rewrite the delta function.

$$\phi(x) = \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3} \int d\omega \left[\frac{\delta(\omega - \sqrt{\vec{k}^2 + m^2})}{|\frac{d}{d\omega}(-\omega^2 + \vec{k}^2 + m^2)|} e^{-i\omega t}\theta(k^0)a(\vec{k}) \right.$$
$$\left. + \frac{\delta(\omega + \sqrt{\vec{k}^2 + m^2})}{|\frac{d}{d\omega}(-\omega^2 + \vec{k}^2 + m^2)|} e^{-i\omega t}\theta(-k^0)a^\dagger(-\vec{k})\right]$$

The denominators evaluates easily, becoming 2ω. All the delta functions tells us is that we're on shell. Doing the integral over ω enforces this condition. We could remove ω, though it's actually easier to stick with our notation (including ω's) and just remember that k is on shell. Hence,

$$\phi(x) = \int \frac{d^3k}{2\omega} \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3}\left[e^{-i\omega t}a(\vec{k}) + e^{i\omega t}a^\dagger(-\vec{k})\right]$$

Distributing:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[e^{ikx} a(\vec{k}) + e^{i\vec{k}\cdot\vec{x}} e^{i\omega t} a^\dagger(-\vec{k})\right]$$

In the second term, we'll rewrite our dummy variable as $k \to -k$. Since we haven't changed the magnitude of the determinant, nothing else will change. Hence,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[e^{ikx} a(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} e^{i\omega t} a^\dagger(\vec{k})\right]$$

Commutation relations don't matter since kx is a Lorentz Invariant. Then,

$$\phi(x) = \int \widetilde{dk} \left[a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx}\right]$$

which is equation 3.19; so we're done, assuming the particle is on shell (which we will always assume, at least for the first several chapters, and certainly for the results we're trying to show).

(c) Derive the first and second claims by taking the Lorentz transformation of the result of part b.

We'll start by assuming that $k^0 \geq 0$. Now take the Lorentz Transformation as indicated, and set aside the components which are already manifestly Lorentz Invariant.

$$U(\Lambda)^{-1} \overline{\phi}(k) U(\Lambda) = 2\pi\delta(k^2 + m^2) \left[U(\Lambda)^{-1} a(k) U(\Lambda)\right] \tag{3.3.1}$$

Now use the result from part a:

$$\overline{\phi}(\Lambda^{-1}k) = 2\pi\delta(k^2 + m^2) U(\Lambda)^{-1} a(k) U(\Lambda)$$

Now use the result from part b again:

$$2\pi\delta((\Lambda^{-1}k)^2 + m^2) a(\Lambda^{-1}k) = 2\pi\delta(k^2 + m^2) U(\Lambda)^{-1} a(k) U(\Lambda)$$

which is:

$$\delta((\Lambda^{-1}k)^2 + m^2) a(\Lambda^{-1}k) = \delta(k^2 + m^2) U(\Lambda)^{-1} a(k) U(\Lambda)$$

Using 2.5, we note that $(\Lambda^{-1}k)^2 = (\Lambda^{-1}k)^\dagger(\Lambda^{-1}k) = k^\dagger(\Lambda^{-1})^\dagger \Lambda^{-1} k = k^\dagger \Lambda\Lambda^{-1} k = k^\dagger k = k^2$. Then,

$$\delta(k^2 + m^2) a(\Lambda^{-1}k) = \delta(k^2 + m^2) U(\Lambda)^{-1} a(k) U(\Lambda)$$

Now we equate the coefficients of the delta functions. This is allowed because two nonequal things cannot become equal when multiplied by a delta function (this is the basis of the delta function expansion, which is at the core of the Green's Functions). Hence,

$$\delta(k^2 + m^2) a(\Lambda^{-1}k) = \delta(k^2 + m^2) U(\Lambda)^{-1} a(k) U(\Lambda)$$

$$a(\Lambda^{-1}k) = U(\Lambda)^{-1} a(k) U(\Lambda)$$

Now we take the Hermitian Conjugates of both sides:

$$a^\dagger(\Lambda^{-1}k) = U(\Lambda)^\dagger a^\dagger(k)U(\Lambda)^{-1\dagger}$$

Remember that the U functions are unitary, so:

$$a^\dagger(\Lambda^{-1}k) = U(\Lambda)^{-1}a^\dagger(k)U(\Lambda)$$

which is the second claim.

What about for $k^0 \leq 0$? Starting with equation (3.3.1), we have exactly the same thing with $a \to a^\dagger$. I don't normally skip steps, but I will do so here since absolutely nothing changes except for the replacement $a \to a^\dagger$. So, we obtain the second result first, taking the Hermitian conjugate to obtain the first result. Hence, we proved the claim for all possible k^0.

(d) Prove the third claim.

Finally, a straightforward proof. Let's write out the left hand side:

$$U(\Lambda)|k_1 \dots k_n\rangle = U(\Lambda)a^\dagger(k_1)\dots a^\dagger(k_n)|0\rangle$$

Inserting a bunch of identities:

$$U(\Lambda)|k_1 \dots k_n\rangle = U(\Lambda)a^\dagger(k_1)U(\Lambda)^{-1}\dots U(\Lambda)a^\dagger(k_n)U(\Lambda)^{-1}U(\Lambda)|0\rangle$$

Using 3.34 rewritten slightly, we have:

$$U(\Lambda)|k_1 \dots k_n\rangle = a^\dagger(\Lambda k_1)\dots a^\dagger(\Lambda k_n)U(\Lambda)|0\rangle$$

$U(\Lambda)$ will obviously not affect the vacuum. Hence,

$$U(\Lambda)|k_1 \dots k_n\rangle = a^\dagger(\Lambda k_1)\dots a^\dagger(\Lambda k_n)|0\rangle$$

$$U(\Lambda)|k_1 \dots k_n\rangle = |\Lambda k_1 \dots \Lambda k_n\rangle$$

Srednicki 3.4. Recall that $\mathbf{T(a)^{-1}\phi(x)T(a) = \phi(x-a)}$, where $\mathbf{T(a) = \exp(-P^\mu a_\mu)}$ is the spacetime translation operator, and $\mathbf{P^0}$ is identified as the hamiltonian H.

(a) Let $\mathbf{a^\mu}$ be infinitesimal, and derive an expression for $[\mathbf{P^\mu}, \boldsymbol{\phi}(\mathbf{x})]$.

We'll start with the given equation:

$$T(a)^{-1}\phi(x)T(a) = \phi(x-a)$$

Making a^μ infinitesimal as suggested:

$$T(\delta a)^{-1}\phi(x)T(\delta a) = \phi(x-\delta a)$$

We'll expand the spacetime translation operator, throwing away all terms higher than first order since a is infinitesimal. On the right, we'll Taylor expand:

$$(1 + iP^\mu a_\mu)\phi(x)(1 - iP^\mu a_\mu) = \phi(x - \delta a)$$

$$\phi(x) + i[P^\mu \delta a_\mu, \phi(x)] = \phi(x) - \delta a_\mu \partial^\mu \phi(x)$$

Hence,

$$i\delta a_\mu [P^\mu, \phi(x)] = -\delta a_\mu \partial^\mu \phi(x)$$

Matching up the coefficients of δa_μ:

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$$

(b) Show that the time component of your result is equivalent to the Heisenberg equation of motion, $\dot{\boldsymbol{\phi}} = \mathbf{i}[\mathbf{H}, \boldsymbol{\phi}]$.

Simply enough, we take our previous result and let $\mu = 0$.

$$[P^0, \phi(x)] = i\partial^0 \phi(x)$$

which is:

$$[H, \phi(x)] = i\dot{\phi}(y)$$

(c) For a free field, use the Heisenberg equation to derive the Klein-Gordon equation.

The Heisenberg equation is:

$$[H(x), \phi(y)] = i\dot{\phi}(y)$$

Using equation 3.25 for $\mathcal{H}$, with $H = \int d^3x \mathcal{H}(x)$:

$$\int d^3x [\frac{1}{2}\Pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2 - \Omega_0, \phi(y)] = i\dot{\phi}(y)$$

Most of the terms commute with $\phi(x)$, only the first remains:

$$\frac{1}{2}\int d^3x [\Pi(x)^2, \phi(y)] = i\dot{\phi}(y)$$

which is:

$$\frac{1}{2}\int d^3x \left(\Pi(x)[\Pi(x), \phi(y)] + [\Pi(x), \phi(y)]\Pi(x)\right) = i\dot{\phi}(y)$$

Inserting 3.28:

$$\frac{1}{2}\int d^3x \left(\Pi(x) i\delta(x-y) + i\delta(x-y)\Pi(x)\right) = i\dot{\phi}(y)$$

which is:

$$i\int d^3x \Pi(x)\delta(x-y) = i\dot{\phi}(y)$$

$$\Pi(y) = \dot{\phi}(y)$$

which is as expected, per equation 3.24. Now we'll use Heisenberg's equation again, this time with $\Pi(y)$ as the field:

$$[H(x), \Pi(y)] = -i\dot{\Pi}(y)$$

Using equation 3.25 for $\mathcal{H}$, with $H = \int d^3x\mathcal{H}(x)$:

$$\int d^3x[\frac{1}{2}\Pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2 - \Omega_0, \phi(y)] = -i\dot{\phi}(y)$$

This time, the middle terms fail to commute:

$$\int d^3x[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2, \Pi(y)] = -i\dot{\Pi}(y)$$

$$\frac{1}{2}\int d^3x \left([(\nabla\phi(x))^2, \Pi(y)] + m^2[\phi(x)^2, \Pi(y)]\right) = -i\dot{\Pi}(y)$$

$$\frac{1}{2}\int d^3x \big(\nabla\phi(x)[\nabla\phi(x), \Pi(y)] + [\nabla\phi(x), \Pi(y)]\nabla\phi(x) + m^2\phi(x)[\phi(x), \Pi(y)]$$
$$+m^2[\phi(x), \Pi(y)]\phi(x)\big) = -i\dot{\Pi}(y)$$

Now, let's note that these derivative operators act only on x, they are "harmless" to functions of y. Hence,

$$\frac{1}{2}\int d^3x \big(\nabla\phi(x)\nabla[\phi(x), \Pi(y)] + \nabla[\phi(x), \Pi(y)]\nabla\phi(x) + m^2\phi(x)[\phi(x), \Pi(y)]$$
$$+m^2[\phi(x), \Pi(y)]\phi(x)\big) = -i\dot{\Pi}(y)$$

Inserting 3.28,

$$\frac{1}{2}\int d^3x \big(\nabla\phi(x)\nabla[i\delta(x-y)] + \nabla[i\delta(x-y)]\nabla\phi(x) + m^2\phi(x)[i\delta(x-y)]$$
$$+m^2[i\delta(x-y)]\phi(x)\big) = -i\dot{\Pi}(y)$$

Now we group some terms. These are all spatial functions of x and y, so commutation is no longer an issue:

$$i\int d^3x \left(\nabla\phi(x)\nabla\delta(x-y)] + m^2\phi(x)[\delta(x-y)]\right) = -i\dot{\Pi}(y)$$

Assuming the field and all its derivatives are bounded at spatial infinity, we can integrate by parts:

$$i\int d^3x \left(-\delta(x-y)\nabla^2\phi(x)] + m^2\phi(x)\delta(x-y)\right) = -i\dot{\Pi}(y)$$

Taking the integral:

$$i\left(-\nabla^2\phi(y) + m^2\phi(y)\right) = -i\dot{\Pi}(y)$$

which is:

$$-\nabla^2\phi(y) + m^2\phi(y) = -\dot{\Pi}(y)$$

On the right side we use equation 3.24 (which we just verified). Let's also switch our independent variable to x, just for cosmetic reasons:

$$\nabla^2\phi(x) - m^2\phi(x) = \ddot{\phi}(x)$$

which is the Klein-Gordon Equation.

(d) Define a spatial momentum operator

$$\mathbf{P} = -\int \mathbf{d^3x}\ \mathbf{\Pi(x)}\boldsymbol{\nabla}\boldsymbol{\phi}\mathbf{(x)}$$

Use the canonical commutation relations to show that P obeys the relation you derived in part a.

What do we expect this to look like? Our result from part a was:

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$$

If we disallow μ from being 0, then we have:

$$[P^i, \phi(x)] = i\partial^i \phi(x)$$

Now we'll write P as a vector of operators and ∂^i as a vector of derivatives:

$$[P, \phi(x)] = i\nabla\phi(x) \tag{3.4.1}$$

Now we're ready to see if the claim comes out to equal this. Let's proceed directly:

$$[P, \phi(y)] = -\int d^3x [\Pi(x)\nabla\phi(x), \phi(y)]$$

which is:

$$[P, \phi(y)] = -\int d^3x [\Pi(x), \phi(x)]\nabla\phi(x)$$

Use 3.28,

$$[P, \phi(y)] = -\int d^3x [-i\delta^3(x-y)]\nabla\phi(x)$$

which is:

$$[P, \phi(y)] = i\int d^3x \delta^3(x-y)\nabla\phi(x)$$

Using the delta function, and switching the independent variable,

$$[P, \phi(x)] = i\nabla\phi(x)$$

which is the same as (3.4.1).

(e) Express P in terms of a(k) and a†(k).

First, we use equation 3.24:

$$P = -\int d^3x\ \dot{\phi}(x)\nabla\phi(x)$$

and now equation 3.19, where we write out the product of the four-vectors:

$$P = -\int d^3x \, \frac{\partial}{\partial t} \left[\int \widetilde{dk} \left(a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} + a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right] \nabla \left[\int \widetilde{dk} \left(a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} + a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right]$$

Now we can take the derivatives:

$$P = -\int d^3x \, \left[\int \widetilde{dk} \left((-i\omega) a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} + i\omega a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right]$$

$$\times \left[\int \widetilde{dk} i\mathbf{k} \left(a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} - i\mathbf{k} a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right]$$

$$P = \int d^3x \, \left[\int \widetilde{dk} \left((-\omega) a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} + \omega a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right] \left[\int \widetilde{dk}\mathbf{k} \left(a(k) e^{-i\omega t + i\mathbf{k}\mathbf{x}} - \mathbf{k} a^\dagger(k) e^{i\omega t - i\mathbf{k}\mathbf{x}} \right) \right]$$

We can simplify a lot if we use two different integration variables for the two different integrals:

$$P = \omega \int d^3x \widetilde{dk} \widetilde{dk'} \, \mathbf{k'} \left[a^\dagger(k) e^{-ikx} - a(k) e^{ikx} \right] \left[a(k') e^{ik'x} - a^\dagger(k') e^{-ik'x} \right]$$

Now we should multiply these out:

$$P = \omega \int d^3x \widetilde{dk} \widetilde{dk'} \, \mathbf{k'} \Big(a^\dagger(k) a(k') e^{i(k'-k)x} - a^\dagger(k) a^\dagger(k') e^{-i(k+k')x} - a(k) a(k') e^{i(k+k')x}$$

$$+ a(k) a^\dagger(k') e^{i(k-k')x} \Big)$$

Now we can do the x integral, using equation 3.27. Note that 3.27 only turns the spatial part into a delta function, the temporal parts remain. However, requiring the spatial parts and the norms to be equal requires the temporal parts to be equal as well. Hence,

$$P = (2\pi)^3 \omega \int \widetilde{dk} \widetilde{dk'} \, \mathbf{k'} \left(a^\dagger(k) a(k') \delta(k' - k) - a^\dagger(k) a^\dagger(k') e^{2i\omega t} \delta(k + k') - a(k) a(k') \delta(k + k') e^{-2i\omega t} \right.$$

$$\left. + a(k) a^\dagger(k') \delta(k - k') \right)$$

Now let's use 3.18 to set up the k' integral:

$$P = (2\pi)^3 \omega \int \widetilde{dk} \frac{d^3k'}{(2\pi)^3 2\omega} \, \mathbf{k'} \left(a^\dagger(k) a(k') \delta(k' - k) - e^{2i\omega t} a^\dagger(k) a^\dagger(k') \delta(k + k') \right.$$

$$\left. - e^{-2i\omega t} a(k) a(k') \delta(k + k') + a(k) a^\dagger(k') \delta(k - k') \right)$$

Now let's do the k' integral:

$$P = (2\pi)^3 \int \widetilde{dk} \frac{1}{(2\pi)^3 2} \, \mathbf{k} \left(a^\dagger(k) a(k) - a^\dagger(k) a^\dagger(-k) e^{2i\omega t} - a(k) a(-k) e^{-2i\omega t} + a(k) a^\dagger(k) \right)$$

The second and third terms are odd (thanks to the $\mathbf{k}$ in front) and vanish when integrating over all space. Then,

$$P = (2\pi)^3 \int \widetilde{dk} \frac{1}{(2\pi)^3 2} \, \mathbf{k} \left(a^\dagger(k) a(k) + a(k) a^\dagger(k) \right)$$

Now we'll use 3.29:

$$P = (2\pi)^3 \int \widetilde{dk} \frac{1}{(2\pi)^3 2} \ \mathbf{k} \left(a^\dagger(k)a(k) + a^\dagger(k)a(k) + (2\pi)^3 2\omega\delta^3(k-k) \right)$$

The delta function is meaningless (technically, it enforces the condition that $0 = 0$, which doesn't need enforcing). So:

$$P = (2\pi)^3 \int \widetilde{dk} \frac{1}{(2\pi)^3} \ \mathbf{k} \left(a^\dagger(k)a(k) + (2\pi)^3\omega \right)$$

This last term is odd, so it vanishes when we integrate over all k:

$$P = \int \widetilde{dk} \ \mathbf{k} a^\dagger(k)a(k)$$

Srednicki 3.5. Consider a complex (that is, nonhermitian) scalar field ϕ with lagrangian density

$$\mathcal{L} = -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi + \Omega_0$$

(a) Show that ϕ obeys the Klein-Gordon Equation.

Start with the given Lagrangian, and integrate to get the action.

$$\int d^4x\mathcal{L} = -\int d^4x \left[\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi + \Omega_0\right]$$

Now require $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, by $\phi \to \phi + \delta\phi$ and $\phi^\dagger \to \phi^\dagger + \delta\phi^\dagger$. The result is:

$$\int d^4x(\mathcal{L} + \delta\mathcal{L}) = -\int d^4x \left[\partial^\mu(\phi^\dagger + \delta\phi^\dagger)\partial_\mu(\phi + \delta\phi) - m^2(\phi^\dagger + \delta\phi^\dagger)(\phi + \delta\phi) + \Omega_0\right]$$

Expanding both sides and throwing away terms higher than first-order:

$$\int d^4x(\mathcal{L}+\delta\mathcal{L}) = -\int d^4x \left[\partial^\mu\phi^\dagger\partial_\mu\phi - \partial^\mu\phi^\dagger\partial_\mu\delta\phi - \partial^\mu\delta\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi - m^2\phi^\dagger\delta\phi - m^2\delta\phi^\dagger\phi + \Omega_0\right]$$

Subtract the Lagrangian from both sides:

$$\int d^4x\delta\mathcal{L} = -\int d^4x \left[\partial^\mu\phi^\dagger\partial_\mu\delta\phi - \partial^\mu\delta\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\delta\phi - m^2\delta\phi^\dagger\phi\right]$$

We use the variational principle to set this equal to zero, then integrate by parts.

$$0 = \int d^4x \left[\partial_\mu\partial^\mu\phi^\dagger\delta\phi + \delta\phi^\dagger\partial^\mu\partial_\mu\phi - m^2\phi^\dagger\delta\phi - m^2\delta\phi^\dagger\phi\right]$$

Now we can kill off the integral (take the derivative of both sides) and factor out the common terms:

$$0 = (\partial_\mu\partial^\mu\phi^\dagger - m^2\phi^\dagger)\delta\phi + (\partial^\mu\partial_\mu\phi - m^2\phi)\delta\phi^\dagger$$

The coefficients of the differentials must be zero, hence:

$$(\partial^2 - m^2)\phi = 0$$
$$(\partial^2 - m^2)\phi^\dagger = 0$$

So both ϕ and $\phi^\dagger$ obey the Klein-Gordon Equation.

(b) Treat ϕ and $\phi^\dagger$ as independent fields, and find the conjugate momentum for each. Compute the hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

The conjugate momentum is:

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Inserting 3.37,

$$\Pi(x) = \frac{\partial}{\partial \dot{\phi}} \left[\dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \nabla\phi - m^2 \phi^\dagger \phi + \Omega_0 \right]$$

$$\Pi(x) = \dot{\phi}^\dagger$$

We do the same thing for the other field:

$$\Pi^\dagger(x) = \frac{\partial}{\partial \dot{\phi}^\dagger} \left[\dot{\phi}^\dagger \dot{\phi} - \nabla\phi^\dagger \nabla\phi - m^2 \phi^\dagger \phi + \Omega_0 \right]$$

$$\Pi^\dagger(x) = \dot{\phi}$$

Now we can construct the Hamiltonian density:

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

Simplifying,

$$\mathcal{H} = \Pi(x)\dot{\phi} + \Pi^\dagger(x)\dot{\phi}^\dagger - \mathcal{L}$$
$$\mathcal{H} = \Pi(x)\Pi^\dagger(x) + \Pi^\dagger(x)\Pi(x) - \mathcal{L}$$
$$\mathcal{H} = \Pi(x)\Pi^\dagger(x) + \Pi^\dagger(x)\Pi(x) - \Pi(x)\Pi^\dagger(x) + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi - \Omega_0$$
$$\mathcal{H} = \Pi^\dagger(x)\Pi(x) + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi - \Omega_0$$

(c) Write the mode expansion of ϕ as

$$\boldsymbol{\phi}(\mathbf{x}) = \int \widetilde{\mathbf{dk}}[\mathbf{a}(\mathbf{k})\mathbf{e}^{\mathbf{ikx}} + \mathbf{b}^\dagger(\mathbf{k})\mathbf{e}^{-\mathbf{ikx}}]$$

Express a(k) and b(k) in terms of ϕ and $\phi^\dagger$ and their time derivative.

We follow the derivation in the text. First, we add some terms to the mode expansion:

$$\int d^3x e^{-ik'x}\phi(x) = \int \widetilde{dk}\ d^3x\ e^{-ik'x}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}]$$

$$\int d^3x e^{-ik'x}\phi(x) = \int \frac{d^3k\ d^3x}{(2\pi)^3 2\omega}\ [a(\mathbf{k})e^{i(k-k')x} + b^\dagger(\mathbf{k})e^{-i(k+k')x}]$$

Now do the spatial integral:

$$\int d^3x e^{-ik'x}\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\ [a(\mathbf{k})e^{-i(\omega-\omega')t}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}') + b^\dagger(\mathbf{k})e^{i(\omega+\omega')t}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')]$$

We simplify. Note that the spatial parts and the norms of k and k' are equal, which implies that the temporal parts must be equal.

$$\int d^3x e^{-ik'x}\phi(x) = \int \frac{d^3k}{2\omega}\ [a(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') + b^\dagger(\mathbf{k})e^{2i\omega t}\delta^3(\mathbf{k}+\mathbf{k}')]$$

which implies:

$$\int d^3x e^{-ik'x}\phi(x) = \frac{1}{2\omega}\ [a(\mathbf{k}') + b^\dagger(-\mathbf{k}')e^{2i\omega t}]$$

Let's replace $k' \to k$ for cosmetic reasons.

$$\int d^3x e^{-ikx}\phi(x) = \frac{1}{2\omega}\ [a(\mathbf{k}) + b^\dagger(-\mathbf{k})e^{2i\omega t}] \tag{3.5.1}$$

Now we repeat this argument, but make the replacement $\phi \to \frac{\partial}{\partial t}\phi$. It is obvious that the first term gains a factor of $-i\omega$ and the second term gains a factor of $i\omega$. Then,

$$\int d^3x e^{-ikx}\partial_0\phi(x) = \frac{i}{2}\ [-a(\mathbf{k}) + b^\dagger(-\mathbf{k})e^{2i\omega t}] \tag{3.5.2}$$

Adding (3.5.1) with (3.5.2) and inserting the necessary constants gives:

$$a(\mathbf{k}) = \int d^3x e^{-ikx}(\omega\phi(x) + i\partial_0\phi(x)) \tag{3.5.3}$$

It seems like we're in a pickle to get $b(\mathbf{k})$, since our formulas so far only give $b(-\mathbf{k})$. Fortunately, if we go back to the mode expansion, we see that switching $\phi \longleftrightarrow \phi^\dagger$ is equivalent to switching $a(\mathbf{k}) \longleftrightarrow b(\mathbf{k})$. We then make this substitution in (3.5.3):

$$b(\mathbf{k}) = \int d^3x e^{-ikx}(\omega\phi^\dagger(x) + i\partial_0\phi^\dagger(x)) \tag{3.5.4}$$

(d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by a(k) and b(k) and their hermitian conjugates.

Let's rewrite equation (3.5.3) using the results from part b:

$$a(\mathbf{k}) = \int d^3x e^{-ikx}(\omega\phi(x) + i\Pi^\dagger(x)) \tag{3.5.5}$$

Similarly for (3.5.4):

$$b(\mathbf{k}) = \int d^3x e^{-ikx}(\omega\phi^\dagger(x) + i\Pi(x)) \tag{3.5.6}$$

We'll start by calculating $[a(k), a(k')]$. We can save ourselves some math by looking at (3.5.6) and noticing that all terms will be proportional to $[\phi, \phi]$, $[\Pi^\dagger, \Pi^\dagger]$, or $[\phi, \Pi^\dagger]$. The canonical commutation relations (eq. 3.28) show that all these terms are zero. Hence, [a, a] = 0, and the hermitian conjugate is also $[a^\dagger, a^\dagger] = 0$.

The same argument will work for the b operator. Hence, $[b, b] = 0$ and $[b^\dagger, b^\dagger] = 0$.

The others we unfortunately have to work through. We'll start with:

$$[a, a^\dagger] = \left[\int d^3x e^{-ikx}(\omega\phi(x) + i\Pi^\dagger(x)), \int d^3x' e^{ikx'}(\omega\phi^\dagger(x') - i\Pi(x'))\right]$$

$$[a, a^\dagger] = \int d^3x d^3x' e^{-i(k-k')x} \left[\omega\phi(x) + i\Pi^\dagger(x), \omega\phi^\dagger(x') - i\Pi(x')\right]$$

$$[a, a^\dagger] = \int d^3x d^3x' e^{-i(k-k')x} \left(-i\omega \left[\phi(x), \Pi(x')\right] - i\omega \left[\phi^\dagger(x'), \Pi^\dagger(x)\right]\right) \tag{3.5.7}$$

$$[a, a^\dagger] = -2i\omega \int d^3x d^3x' e^{-i(k-k')x} \left[\phi(x), \Pi(x')\right]$$

Using 3.28:

$$[a, a^\dagger] = -2i\omega \int d^3x d^3x' e^{-i(k-k')x} i\delta^3(x - x')$$

$$[a, a^\dagger] = 2\omega \int d^3x e^{-i(k-k')x}$$

$$[a, a^\dagger] = (2\pi)^3 2\omega\delta^3(\mathbf{k} - \mathbf{k}')$$

where, as usual, the temporal parts must be equal (and therefore cancel) since the norms and the spatial parts are equal.

For $[b, b^\dagger]$, we notice that the same argument will work with $\Pi \longleftrightarrow \Pi^\dagger$ and $\phi \longleftrightarrow \phi^\dagger$. At equation (3.5.7) this substitution becomes trivial, and the result is the same. Hence,

$$[b, b^\dagger] = (2\pi)^3 2\omega\delta^3(\mathbf{k} - \mathbf{k}')$$

Next,

$$[a, b] = [\int d^3x e^{-ikx}(\omega\phi(x) + i\Pi^\dagger(x)), \int d^3x' e^{-ikx'}(\omega\phi^\dagger(x') + i\Pi(x'))]$$

$$[a, b] = \int d^3x d^3x' e^{-ik(x+x')}[\omega\phi(x) + i\Pi^\dagger(x), \omega\phi^\dagger(x') + i\Pi(x')]$$

$$[a, b] = i\omega \int d^3x d^3x' e^{-ik(x+x')} \left([\phi(x), \Pi(x')] - [\phi^\dagger(x'), \Pi^\dagger(x)]\right)$$

$$[a, b] = i\omega \int d^3x d^3x' e^{-ik(x+x')} \left([\phi(x), \Pi(x')] + [\phi(x'), \Pi(x)]^\dagger\right)$$

$$[a, b] = i\omega \int d^3x d^3x' e^{-ik(x+x')} \left(i\delta^3(x - x') - i\delta^3(x - x')\right)$$

$$[a, b] = 0$$

Taking the Hermitian conjugate,

$$[a^\dagger, b^\dagger] = 0$$

The last relationship to verify involves:

$$[a, b^\dagger] = [\int d^3x e^{-ikx}(\omega\phi(x) + i\Pi^\dagger(x)), \int d^3x e^{ikx}(\omega\phi(x) + i\Pi^\dagger(x))]$$

This simplifies to:

$$[a, b^\dagger] = \int d^3x d^3x' e^{-ik(x-x')}[\omega\phi(x) + i\Pi^\dagger(x), \omega\phi(x') + i\Pi^\dagger(x')]$$

All of these terms commute with each other. Hence,

$$[a, b^\dagger] = 0$$

The Hermitian conjugate is:

$$[a^\dagger, b] = 0$$

(e) Express the hamiltonian in terms of a(k) and b(k) and their hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

$$H = \int d^3x \mathcal{H}$$

Now using the result from b,

$$H = \int d^3x \left[\Pi^\dagger(x)\Pi(x) + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi - \Omega_0\right]$$

We'll insert the mode expansion of ϕ. We can also integrate the last term trivially.

$$H = -\Omega_0 V + \int d^3x \left[\int \widetilde{dk}\frac{\partial}{\partial t}[a(k)e^{ikx} + b^\dagger(k)e^{-ikx}] \int \widetilde{dk}\frac{\partial}{\partial t}[a^\dagger(k)e^{-ikx} + b(k)e^{ikx}]\right.$$

$$+ \int \widetilde{dk}\widetilde{dk'}\nabla[a^\dagger(k)e^{-ikx} + b(k)e^{ikx}]\nabla[a(k')e^{ik'x} + b^\dagger(k')e^{-ik'x}]$$

$$\left. + m^2 \int \widetilde{dk}\widetilde{dk'}[a^\dagger(k)e^{-ikx} + b(k)e^{ikx}][a(k')e^{ik'x} + b^\dagger(k')e^{-ik'x}]\right]$$

We'll clean this up a little bit and evaluate some derivatives:

$$H = -\Omega_0 V + \int d^3x\widetilde{dk}\widetilde{dk'} \left([(-i\omega)a(k)e^{ikx} + b^\dagger(k)(i\omega)e^{-ikx}][a^\dagger(k')(i\omega)e^{-ik'x} + b(k')(-i\omega)e^{ik'x}]\right)$$

$$+ \int d^3x\widetilde{dk}\widetilde{dk'} \left([a^\dagger(k)(-ik)e^{-ikx} + b(k)(ik)e^{ikx}][a(k')(ik')e^{ik'x} + b^\dagger(k')(-ik')e^{-ik'x}]\right)$$

$$+ m^2 \int d^3x\widetilde{dk}\widetilde{dk'}[a^\dagger(k)e^{-ikx} + b(k)e^{ikx}][a(k')e^{ik'x} + b^\dagger(k')e^{-ik'x}]$$

We clean up further:

$$H = -\Omega_0 V - \omega^2 \int d^3x\widetilde{dk}\widetilde{dk'} \Big(-a(k)a^\dagger(k')e^{-i(k'-k)x} + a(k)b(k')e^{i(k'+k)x} + b^\dagger(k)a^\dagger(k')e^{-i(k+k')x}$$

$$-b^\dagger(k)b(k')e^{i(k'-k)x}\Big) - kk' \int d^3x\widetilde{dk}\widetilde{dk'} \Big(-a^\dagger(k)a(k')e^{i(k'-k)x} + a^\dagger(k)b^\dagger(k')e^{-i(k+k')x}$$

$$+b(k)a(k')e^{i(k+k')x} - b(k)b^\dagger(k')e^{-i(k'-k)x}\Big) + m^2 \int d^3x\widetilde{dk}\widetilde{dk'} \Big(a^\dagger(k)a(k')e^{i(k'-k)x}$$

$$+a^\dagger(k)b^\dagger(k')e^{-i(k+k')x} + b(k)a(k')e^{i(k+k')x} + b(k)b^\dagger(k')e^{-i(k'-k)x}\Big)$$

Now we take the x-integrals. We'll also rewrite the k' integral, and we set $\omega = \omega'$ since the norms and spatial parts are equal.

$$H = -\Omega_0 V - \omega^2(2\pi)^3 \int \widetilde{dk}\frac{d^3k'}{(2\pi)^3 2\omega'} \big(-a(k)a^\dagger(k')\delta^3(k'-k) + a(k)b(k')\delta^3(k'+k)e^{-2i\omega t}$$

$$+b^\dagger(k)a^\dagger(k')\delta^3(k+k')e^{2i\omega t} - b^\dagger(k)b(k')\delta^3(k'-k)\big) - kk'(2\pi)^3 \int \widetilde{dk}\frac{d^3k'}{(2\pi)^3 2\omega'} \big(-a^\dagger(k)a(k')\delta^3(k'-k)$$

$$+a^\dagger(k)b^\dagger(k')\delta^3(k'+k)e^{2i\omega t} + b(k)a(k')\delta^3(k'+k)e^{-2i\omega t} - b(k)b^\dagger(k')\delta^3(k'-k)\big)$$

$$+m^2(2\pi)^3 \int \widetilde{dk}\frac{d^3k'}{(2\pi)^3 2\omega'} \big(a^\dagger(k)a(k')\delta^3(k'-k) + a^\dagger(k)b^\dagger(k')\delta^3(k'+k)e^{2i\omega t} + b(k)a(k')\delta^3(k+k')e^{-2i\omega t}$$

$$+b(k)b^\dagger(k')\delta^3(k'-k)\big)$$

Now we do the k' integral:

$$H = -\Omega_0 V - \omega^2 \int \widetilde{dk}\frac{1}{2\omega'} \big(-a(k)a^\dagger(k) + a(k)b(-k)e^{-2i\omega t} + b^\dagger(k)a^\dagger(-k)e^{2i\omega t} - b^\dagger(k)b(k)\big)$$

$$-kk' \int \widetilde{dk}\frac{d^3k'}{2\omega'} \big(-a^\dagger(k)a(k) - a^\dagger(k)b^\dagger(-k)e^{2i\omega t} - b(k)a(-k)e^{-2i\omega t} - b(k)b^\dagger(k)\big)$$

$$+m^2 \int \widetilde{dk}\frac{1}{2\omega'} \big(a^\dagger(k)a(k) + a^\dagger(k)b^\dagger(-k)e^{2i\omega t} + b(k)a(-k)e^{-2i\omega t} + b(k)b^\dagger(k)\big)$$

Let's next simplify:

$$H = -\Omega_0 V + \frac{1}{2\omega}\Big\{-\omega^2 \int \widetilde{dk} \big(-a(k)a^\dagger(k) + a(k)b(-k)e^{-2i\omega t} + b^\dagger(k)a^\dagger(-k)e^{2i\omega t} - b^\dagger(k)b(k)\big)$$

$$+k^2 \int \widetilde{dk} \big(a^\dagger(k)a(k) + a^\dagger(k)b^\dagger(-k)e^{2i\omega t} + b(k)a(-k)e^{-2i\omega t} + b(k)b^\dagger(k)\big)$$

$$+m^2 \int \widetilde{dk} \big(a^\dagger(k)a(k) + a^\dagger(k)b^\dagger(-k)e^{2i\omega t} + b(k)a(-k)e^{-2i\omega t} + b(k)b^\dagger(k)\big)\Big\}$$

These last two terms are the same up to a constant, so let's consolidate:

$$H = -\Omega_0 V + \frac{1}{2\omega}\Big\{\omega^2 \int \widetilde{dk} \big(a(k)a^\dagger(k) - b^\dagger(k)a^\dagger(-k)e^{2i\omega t} - a(k)b(-k)e^{-2i\omega t} + b^\dagger(k)b(k)\big)$$

$$+(k^2+m^2)\int \widetilde{dk}\left(a^\dagger(k)a(k)+a^\dagger(k)b^\dagger(-k)e^{2i\omega t}+b(k)a(-k)e^{-2i\omega t}+b(k)b^\dagger(k)\right)\Big\}$$

Now we use the commutation relations to simplify:

$$H=-\Omega_0 V+\frac{1}{2\omega}\Big\{\omega^2\int \widetilde{dk}\left(a^\dagger(k)a(k)-a^\dagger(-k)b^\dagger(k)e^{2i\omega t}-b(-k)a(k)e^{-2i\omega t}\right.$$

$$\left.+b(k)b^\dagger(k)\right)+(k^2+m^2)\int \widetilde{dk}\left(a^\dagger(k)a(k)+a^\dagger(k)b^\dagger(-k)e^{2i\omega t}+b(k)a(-k)e^{-2i\omega t}\right.$$

$$\left.+b(k)b^\dagger(k)\right)\Big\}$$

Rewriting, and negating the integration variable when needed,

$$H=-\Omega_0 V+\frac{1}{2\omega}\Big\{(\omega^2+k^2+m^2)\int \widetilde{dk}\left(a^\dagger(k)a(k)+b(k)b^\dagger(k)\right)$$

$$+(-\omega^2+k^2+m^2)\int \widetilde{dk}\left(a^\dagger(-k)b^\dagger(k)e^{2i\omega t}-b(-k)a(k)e^{-2i\omega t}\right)\Big\}$$

The second term vanishes since the particles are on shell. Simplifying the first term gives:,

$$H=-\Omega_0 V+\frac{1}{2\omega}\Big\{2\omega^2\int \widetilde{dk}\left(a^\dagger(k)a(k)+b(k)b^\dagger(k)\right)\Big\}$$

$$H=-\Omega_0 V+\omega\Big\{\int \widetilde{dk}\left(a^\dagger(k)a(k)+b(k)b^\dagger(k)\right)\Big\}$$

Commute this last term:

$$H=-\Omega_0 V+\omega\Big\{\int \widetilde{dk}\left(a^\dagger(k)a(k)+b^\dagger(k)b(k)+(2\pi)^3 2\omega\delta^3(0)\right)\Big\}$$

Interpreting $(2\pi)^3\delta^3(0)$ as the volume of space (as in the text),

$$H=-\Omega_0 V+\omega\Big\{\int \widetilde{dk}\left(a^\dagger(k)a(k)+b^\dagger(k)b(k)+2\omega V\right)\Big\}$$

Using equation 3.29 and the definition of $\widetilde{dk}$, we have:

$$H=(2\mathcal{E}_0-\Omega_0)V+\int \widetilde{dk}\;\omega\left[a^\dagger(k)a(k)+b^\dagger(k)b(k)\right]$$

$\mathcal{E}_0$ represents the zero point energy for each set of oscillators. To set the ground state energy equal to zero, we must choose $\Omega_0=2\mathcal{E}_0$