Srednicki Chapter 29

QFT Problems & Solutions

A. George

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Srednicki 29.1. Consider a theroy with a single dimensionless coupling g whose beta function takes the form $\beta(g) = b_1 g^2 + b_2 g^3 + \ldots$. Now consider a new definiton of the coupling $\tilde{g}$ that agrees with the original definition at lowest order, so that we have $\tilde{g} = g + c_2 g^2 + \ldots$.

(a) Show that $\beta(\tilde{g}) = b_1 \tilde{g}^2 + b_2 \tilde{g}^3$.

We have:

$$g = \tilde{g} - c_2 g^2 - \ldots$$

Using this in our beta function, we have:

$$\beta(\tilde{g}) = b_1(\tilde{g} - c_2 g^2 - \ldots)^2 + b_2(\tilde{g} - c_2 g^2 - \ldots)^3 + \ldots$$

Hence:

$$\beta(\tilde{g}) = b_1 \tilde{g}^2 - 2b_1 c_2 g^2 \tilde{g} + b_1 c_2 g^4 + \ldots + b_2 \tilde{g}^3 - 2b_2 c_2 \tilde{g}^2 g^2 + \ldots$$

We see that the minimum order in g is the minimum order in $\tilde{g}$. So we neglect all terms where $O(g) + O(\tilde{g}) > 3$. Then:

$$\beta(\tilde{g}) = b_1 \tilde{g}^2 - 2b_1 c_2 g^2 \tilde{g} + b_2 \tilde{g}^3 + \ldots$$

The second term has two coefficients, each of which should be negligable (this is the idea of a perturbation sequence: the first terms are the most important). Thus,

$$\beta(\tilde{g}) = b_1 \tilde{g}^2 + b_2 \tilde{g}^3 + \ldots$$

as expected.

(b) Generalize this result to the case of multiple dimensionless couplings.

As before, we have:

$$g = \tilde{g} - c_2 g^2 - d_2 gh - e_3 h^2 - \ldots$$

where g and h are the dimensionless constants (additional dimensionless constants can be added, with all possible permutations up to order two in the constants). The beta function (shown for two constants, but easy to extend to n constants) is:

$$\beta(\tilde{g}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\left[\tilde{g} - (c_2g^2 + d_2gh + e_3h^2 + \ldots)\right]^m \left[\tilde{h} - (c_2g^2 + d_2gh + e_3h^2 + \ldots)\right]^n$$

Dropping all terms that have more than one coefficient:

$$\beta(\tilde{g}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\tilde{g}^m\tilde{h}^n + \ldots$$

as expected.

Srednicki 29.2. Consider ϕ^3 theory in six euclidean spacetime dimensions, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}Z(\Lambda_0)\partial_\mu\phi\partial_\mu\phi + \frac{1}{24}Z^{3/2}(\Lambda_0)g(\Lambda_0)\phi^3$$

We assume that we have fine-tuned to keep $m^2(\Lambda) \ll \Lambda^2$, and so we neglect the mass term.

(a) Show that

$$Z(\Lambda) = Z(\Lambda_0)\left(1 - \frac{1}{2}g^2(\Lambda_0)\frac{d}{dk}\left[\int_\Lambda^{\Lambda_0}\frac{d^6\ell}{(2\pi)^6}\frac{1}{(k+\ell)^2\ell^2}\right]_{k^2=0} + \ldots\right)$$

$$g(\Lambda) = \frac{Z^{3/2}(\Lambda_0)}{Z^{3/2}(\Lambda)}g(\Lambda_0)\left(1 + g^2(\Lambda_0)\int_\Lambda^{\Lambda_0}\frac{d^6\ell}{(2\pi)^6}\frac{1}{(\ell^2)^3} + \ldots\right)$$

Hint: note that the tree-level propagator is $\tilde{\Delta}(k) = [Z(\Lambda_0)k^2]^{-1}$

This is a very difficult (and poorly-explained) problem. Let's clarify exactly what we're supposed to do. The Lagrangian presented is given in terms of parameters that are well-defined for an ultraviolet cutoff of Λ_0. Now we want to lower the ultraviolet cutoff to Λ, and quantify the effect on Z and g.

First, we use the hint to observe that:

$$Z(\Lambda_0) = \frac{d}{dk^2}[\tilde{\Delta}(k,\Lambda_0)]^{-1}\bigg|_{k^2=0} \tag{29.2.1}$$

(of course, there is no need in this form to set $k = 0$, but it doesn't hurt to do so, and it will make life simpler later on).

This still holds at the revised energy scale, so:

$$Z(\Lambda) = \frac{d}{dk^2}[\tilde{\Delta}(k,\Lambda)]^{-1}\bigg|_{k^2=0} \tag{29.2.2}$$

Switching gears for a moment, we notice that the propagator at the new scale ($\tilde{\Delta}(\Lambda)$) equals the propagator at the old scale ($\tilde{\Delta}(\Lambda_0)$) plus corrections to the old scale (the corrections being loop diagrams, with momenta between Λ and Λ_0). Thus, the idea behind equation 14.16 holds. Manipulating equation 14.16, we have:

$$\tilde{\Delta}(\Lambda, k)^{-1} = \tilde{\Delta}(\Lambda_0, k)^{-1} - \Pi(\Lambda, \Lambda_0, k)$$

Now we take the derivative of this with respect to k^2, and set $k^2 = 0$ after that. Using (29.2.1) and (29.2.2), this becomes:

$$Z(\Lambda) = Z(\Lambda_0) - \frac{d}{dk^2}\Pi(k^2, \Lambda_0, \Lambda) \tag{29.2.3}$$

Now we need to know Π in order to make further progress. The diagrams are just the tree level diagram, and the one-loop diagram from figure 14.1. The tree-level diagram has a value of 1. The loop diagram gets:

- $\frac{1}{2}$, a symmetry factor
- $\left(-Z(\Lambda_0)g(\Lambda_0)\right)^2$, the vertex factors. Notice that we get a negative sign rather than a factor of i since we are in Euclidean coordinates.
- $\int_\Lambda^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\frac{1}{k^2(k+\ell)^2}$, the propagators from the internal scalars (no mass terms since we neglected those in the Lagrangian).

Due to the Euclidean space, these construct Π, not $i\Pi$.

Putting these together, and using equation (29.2.3), we have:

$$\boxed{Z(\Lambda) = Z(\Lambda_0)\left[1 - \frac{1}{2}g(\Lambda_0)^2\frac{d}{dk^2}\int_\Lambda^{\Lambda_0}\frac{d^6\ell}{(2\pi)^6}\frac{1}{k^2(k+\ell)^2}\bigg|_{k^2=0}\right]}$$

Now for the $g(\Lambda)$ expression. We need the exact form for the vertex. From the Lagrangian, it is clear that this will be $-Z(\Lambda)^{3/2}g(\Lambda)$. Now we just need to draw the diagrams that equal this vertex. We have the tree-level diagram and the one-loop diagram drawn in figure 16.1. Assessing the values of these diagrams in the usual way (there are no factors of i due to the Euclidean space, and no masses since we neglected those terms), we have:

$$-Z^{3/2}(\Lambda)g(\Lambda) = -Z^{3/2}(\Lambda_0)g(\Lambda_0) - Z^{9/2}(\Lambda_0)g^3(\Lambda_0)\int_\Lambda^{\Lambda_0}\frac{d^6\ell}{(2\pi)^6}\frac{1}{\ell^6 Z(\Lambda_0)^3}$$

Solving this, we have:

$$\boxed{g(\Lambda) = \frac{Z^{3/2}(\Lambda_0)}{Z^{3/2}(\Lambda)}g(\Lambda_0)\left[1 + g^2(\Lambda_0)\int_\Lambda^{\Lambda_0}\frac{d^6\ell}{(2\pi)^6}\frac{1}{\ell^6}\right]} \tag{29.2.4}$$

(b) Use your results to compute the beta function

$$\boldsymbol{\beta(g(\Lambda)) = \frac{d}{d\ln\Lambda}g(\Lambda)}$$

and compare with the beta function in section 27.

We need to start by cleaning up $Z(\Lambda)$. We have:

$$Z(\Lambda) = Z(\Lambda_0)\left[1 - \frac{1}{2}g(\Lambda_0)^2 \frac{d}{dk^2}\int_{\Lambda_0}^{\Lambda} \frac{d^6\ell}{(2\pi)^6}\frac{1}{k^2(k+\ell)^2}\Bigg|_{k^2=0}\right]$$

Now let's focus for the moment on this part:

$$\frac{d}{dk^2}\int_{\Lambda}^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\frac{1}{k^2(k+\ell)^2}\Bigg|_{k^2=0}$$

Notice that this is equation 14.12, with $m = 0$. Using the result, we have:

$$\frac{d}{dk^2}\int_{\Lambda}^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\int_0^1 dx\left[q^2 + x(1-x)k^2\right]^{-2}_{k^2=0}$$

where $q = \ell + kx$. Now we take as an approximation that $q \approx \ell$; this is certainly almost true, since $\ell \approx \Lambda$, the very high ultraviolet cutoff. Then, we take the derivative:

$$-2\int_{\Lambda}^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\int_0^1 dx\frac{x(1-x)}{[q^2 + x(1-x)k^2]^3}\Bigg|_{k^2=0}$$

Now we set $k^2 = 0$, and we have:

$$-2\int_{\Lambda}^{\Lambda_0} \frac{d^6\ell}{(2\pi)^6}\int_0^1 dx\frac{x(1-x)}{\ell^6}$$

Now we need to do that ℓ integral. We divide this into angular and radial parts; the angular parts can be done with equation 14.23, in d=6. The radial part can be done directly, as can the x-integral. The result is:

$$Z(\Lambda) = Z(\Lambda_0)\left[1 + \frac{g(\Lambda_0)^2}{384\pi^3}\log\left(\frac{\Lambda_0}{\Lambda}\right)\right] \qquad (29.2.5)$$

Now we need a clean expression for $g(\Lambda)$. In equation (29.2.4), we can drop the second term and take, at first order, $Z(\Lambda) = Z(\Lambda_0)$ from equation (29.2.5) to obtain:

$$g(\Lambda) = g(\Lambda_0) + \ldots$$

This will yield a trivial beta function, however. To the next highest order, we can do two things: we can keep the second term in equation (29.2.4), or we can expand Z to second-order in equation (29.2.5). Doing both will yield something of even higher order, which we are not interested in. Let's start with expanding Z. We then have:

$$\frac{Z^{3/2}(\Lambda_0)g(\Lambda_0)}{Z^{3/2}(\Lambda)} = \frac{Z^{3/2}(\Lambda_0)g(\Lambda_0)}{Z^{3/2}(\Lambda_0)\left[1 + \frac{g(\Lambda_0)^2}{384\pi^3}\log(\Lambda_0/\Lambda)\right]^{3/2}}$$

which is:

$$-\frac{3}{2}\frac{g(\Lambda_0)^3}{384\pi^3}\log(\Lambda_0/\Lambda)$$

Good. Now we can add the second-order term from (29.2.4). We do the integral as before, with the help of equation 14.23, and we get a contribution of:

$$\frac{g(\Lambda_0)^3}{64\pi^3}\log(\Lambda_0/\Lambda)$$

adding these terms up, we have:

$$g(\Lambda) = g(\Lambda_0) + \frac{3}{4}\frac{g(\Lambda_0)^3}{64\pi^3}\log\left(\frac{\Lambda_0}{\Lambda}\right)$$

and finally we can calculate the derivate:

$$\boxed{\beta = \frac{dg(\Lambda)}{d\log\Lambda} = -\frac{3}{4}\frac{g(\Lambda_0)^3}{64\pi^3}}$$

How does this compare with the result from chapter 27? We had there:

$$\frac{d\alpha}{d\log\mu} = -\frac{3}{2}\alpha^2$$

$$\Longrightarrow \frac{d\alpha}{dg}\frac{dg}{d\log\mu} = -\frac{3}{2}\frac{g^4}{(64\pi^3)^2}$$

$$\Longrightarrow \frac{dg}{d\log\mu} = -\frac{3}{4}\frac{g(\Lambda_0)^3}{64\pi^3}$$

which is the same as our solution.

Note: This is by far the most confusing problem so far, and it is a pity that no discussion is included. Srednicki's solution of this problem leaves a lot to be desired; in particular, the comparison with the book is downright wrong. The approximation the $q \approx \ell$ also seems problematic to me, but this must be acceptable since the beta function works out. The key point that really should be noted is that this problem is an excellent illustration of what beta functions are – beta functions show how coupling constants vary with energy. In this problem, we changed the energy, so the degree of change had better reflect the beta function. The agreement with chapter 27 is therefore not coincidental, but absolutely crucial.