Srednicki Chapter 28

QFT Problems & Solutions

A. George

December 12, 2012

Srednicki 28.1. Consider ϕ^4 theory,

$$\boldsymbol{\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - \frac{1}{24}Z_\lambda\lambda\tilde{\mu}^\varepsilon\phi^4}$$

in $d = 4 - \varepsilon$ dimensions. Compute the beta function to $O(\lambda^2)$, the anomalous dimension of m to $O(\lambda)$, and the anomalous dimension of ϕ to $O(\lambda)$.

Let's compute the beta function "from scratch." After that, we'll use the equations that Srednicki derived. We begin with the renormalized Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - \frac{1}{24}Z_\lambda\lambda\tilde{\mu}^\varepsilon\phi^4$$

and the Lagrangian for bare fields:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{24}\lambda_0\tilde{\mu}^\varepsilon\phi_0^4$$

The unique way to relate the bare fields to the renormalized fields is via:

$$\phi_0 = Z_\phi^{1/2}\phi$$

$$m_0 = Z_\phi^{-1/2}Z_m^{1/2}m$$

$$\lambda_0 = Z_\phi^{-2}Z_\lambda\tilde{\mu}^\varepsilon\lambda \qquad (28.1.1)$$

Equation (28.1.1) implies that:

$$\ln\lambda_0 = \ln(Z_\phi^{-2}Z_\lambda) + \varepsilon\ln(\tilde{\mu}) + \ln(\lambda)$$

Defining $G(\lambda, \tilde{\mu})$:

$$\ln\lambda_0 = G(\lambda,\tilde{\mu}) + \varepsilon\ln(\tilde{\mu}) + \ln(\lambda) \qquad (28.1.2)$$

where

$$G(\lambda,\tilde{\mu}) = \ln(Z_\phi^{-2}Z_\lambda) \qquad (28.1.3)$$

Now the result from problem 14.5:

$$Z_\phi = 1 + O(\lambda^2) \qquad (28.1.4)$$

and the result from problem 16.1:

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] + O(\lambda^2) \tag{28.1.5}$$

Now it is necessary to adjust equation (28.1.5) in the $\overline{\text{MS}}$ renormalization scheme, which gives:

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2\varepsilon} + O(\lambda^2) \tag{28.1.6}$$

Now we use equations (28.1.4) and (28.1.6) in equation (28.1.3), giving:

$$G(\lambda, \tilde{\mu}) = \ln\left(1 + \frac{3\lambda}{16\pi^2\varepsilon} + O(\lambda^2)\right) \tag{28.1.7}$$

From the general structure of the Zs, we can write $G = \sum_{n=1}^{\infty} \frac{G_n}{n}$. Expanding equation (28.1.7) and matching the terms, we're left with $G = \frac{3\lambda}{16\pi^2\varepsilon} + O(\varepsilon^{-2})$. Putting this into (28.1.2), we have:

$$\ln\lambda_0 = \frac{3\lambda}{16\pi^2\varepsilon} + O(\varepsilon^{-2}) + O(\lambda^2) + \varepsilon\ln(\tilde{\mu}) + \ln(\lambda)$$

Neglecting higher-order terms and taking the derivative, we have:

$$\frac{d\ln\lambda_0}{d\ln\mu} = \frac{3}{16\pi^2\varepsilon}\frac{d\lambda}{d\ln\mu} + \varepsilon\frac{d\ln\tilde{\mu}}{d\ln\mu} + \frac{1}{\lambda}\frac{d\lambda}{d\ln\mu}$$

λ_0 should be independent of μ. Further, we calculate that the derivative in the second term is one. Thus:

$$0 = \varepsilon + \left(\frac{1}{\lambda} + \frac{3}{16\pi^2\varepsilon}\right)\frac{d\lambda}{d\ln\mu}$$

Multiplying by λ:

$$0 = \varepsilon\lambda + \left(1 + \frac{3\lambda}{16\pi^2\varepsilon}\right)\frac{d\lambda}{d\ln\mu}$$

This is:

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon\lambda\left(1 + \frac{3\lambda}{16\pi^2\varepsilon}\right)^{-1}$$

Expanding in the small λ limit:

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon\lambda\left(1 - \frac{3\lambda}{16\pi^2\varepsilon}\right)$$

which is:

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon\lambda + \frac{3\lambda^2}{16\pi^2}$$

Now we match terms with equation 28.20 (our derivation is still "from scratch," since equation 28.20 stands on its own, see the preceeding paragraph in Srednicki for the justification). The zero-order term (in ε) determines the beta function. Thus,

$$\boxed{\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)}$$

Note that this is the same result we get using equation 28.21 – provided that we determine G correctly. Equation 28.14 is not a general result, however.

Equation (28.1.1) tells us that:

$$M = \ln\left(Z_\phi^{-1/2} Z_m^{1/2}\right)$$

As discussed above, Z_ϕ does not contribute. Z_m was calculated in problem 14.5, we must adjust the result to the $\overline{\text{MS}}$ renormalization scheme, which gives:

$$Z_m = 1 + \frac{\lambda}{16\pi^2\varepsilon}$$

Thus,

$$M = \ln\left([1 + \frac{\lambda}{16\pi^2\varepsilon}]^{1/2}\right)$$

Expanding the radical:

$$M = \ln\left(1 + \frac{\lambda}{32\pi^2\varepsilon}\right)$$

Expanding the natural log:

$$M = \frac{\lambda}{32\pi^2\varepsilon} + O(\varepsilon^{-2})$$

Thus:

$$M_1 = \frac{\lambda}{32\pi^2}$$

and according to equation 28.29 (line 1):

$$\boxed{\gamma_m(\lambda) = \frac{\lambda}{32\pi^2} + O(\lambda^2)}$$

Finally, equation (28.1.1) gives us:

$$a = Z_\phi^{1/2}$$

As discussed above, Z_ϕ does not contribute to order λ. Thus,

$$a = O(\lambda^2)$$

which implies:

$$\boxed{\gamma_\phi(\lambda) = O(\lambda^2)}$$

Srednicki 28.2. Repeat problem 28.1 for the theory of problem 9.3

The renormalized Lagrangian in question is:

$$\mathcal{L} = -Z_\phi \partial^\mu \phi^\dagger \partial_\mu \phi - Z_m m^2 \phi^\dagger \phi - \frac{1}{4} Z_\lambda \lambda (\phi^\dagger \phi)^2$$

The bare-field Lagrangian is:

$$\mathcal{L} = -\partial^\mu \phi_0^\dagger \partial_\mu \phi_0 - m_0^2 \phi_0^\dagger \phi_0 - \frac{1}{4} \lambda_0 (\phi_0^\dagger \phi_0)^2$$

Matching up the terms, we have:

$$\phi_0 = Z_\phi^{1/2}\phi$$
$$m_0 = Z_\phi^{-1/2} Z_m^{1/2} m$$
$$\lambda_0 = Z_\phi^{-2} Z_\lambda \lambda$$

This gives:

$$G = \ln\left(Z_\phi^{-2} Z_\lambda\right)$$
$$M = \ln\left(Z_\phi^{-1/2} Z_m^{1/2}\right)$$
$$a = \ln\left(Z_\phi^{1/2}\right)$$

Problems 14.6 and 16.2 give, after adjusting for the $\overline{\text{MS}}$ renormalization scheme:

$$Z_\phi = 1 + O(\lambda^2)$$
$$Z_m = 1 + \frac{\lambda}{8\pi^2\varepsilon} + O(\lambda^2)$$
$$Z_\lambda = 1 + \frac{5\lambda}{16\pi^2\varepsilon} + O(\lambda^2)$$

Thus,

$$G = \ln\left(1 + \frac{5\lambda}{16\pi^2\varepsilon} + O(\lambda^2)\right)$$
$$M = \ln\left(\left[1 + \frac{\lambda}{8\pi^2\varepsilon} + O(\lambda^2)\right]^{1/2}\right)$$
$$a = \ln\left(1 + O(\lambda^2)\right)$$

Expanding, we find:

$$G_1 = \frac{5\lambda}{16\pi^2}$$
$$M_1 = \frac{\lambda}{16\pi^2}$$
$$a_1 = 0$$

Using 28.21, 28.29, and 28.37, we find:

$$\beta(\lambda) = \frac{5\lambda^2}{16\pi^2} + O(\lambda^3)$$
$$\gamma_m(\lambda) = \frac{\lambda}{16\pi^2} + O(\lambda^2)$$
$$\gamma_\phi(\lambda) = O(\lambda^2)$$

Srednicki 28.3. Consider the lagrangian density:

$$\boldsymbol{\mathcal{L} = -\frac{1}{2} Z_\phi \partial^\mu\phi\partial_\mu\phi - \frac{1}{2} Z_m m^2\phi^2 + Y\phi - \frac{1}{2} Z_\chi \partial^\mu\chi\partial_\mu\chi - \frac{1}{2} Z_M M^2\chi^2}$$

$$+\frac{1}{6}Z_g g\tilde{\mu}^{\varepsilon/2}\phi^3+\frac{1}{2}Z_h h\tilde{\mu}^{\varepsilon/2}\phi\chi^2$$

in $d=6-\varepsilon$ dimensions, where ϕ and χ are real scalar fields, and Y is adjusted to make $\langle 0|\phi(x)|0\rangle = 0$. (Why is no such contribution needed for χ?)

The Y term is needed to cancel the tadpole diagrams, those diagrams with only one "source," ie external line. It is impossible to draw a diagram with only one external χ line, since χ enters the Lagrangian only in even powers.

(a) Compute the one-loop contributions to each of the Zs in the $\overline{\text{MS}}$ renormalization scheme

This is an excellent review question, so let's work through this slowly and in detail. Let's use the solid line for ϕ particles and the dashed line for χ particles.

We can write the Lagrangian as $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_{ct}$, where:

$$\mathcal{L}_0=-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2-\frac{1}{2}\partial^\mu\chi\partial_\mu\chi-\frac{1}{2}M^2\chi^2$$

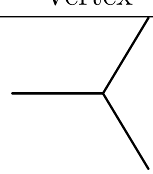
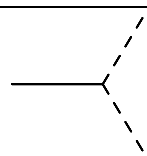
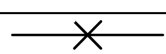
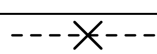
$$\mathcal{L}_1=\frac{1}{6}Z_g g\tilde{\mu}^{\varepsilon/2}\phi^3+\frac{1}{2}Z_h h\tilde{\mu}^{\varepsilon/2}\phi\chi^2$$

$$\mathcal{L}_{ct}=-\frac{1}{2}(Z_\phi-1)\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}(Z_m-1)m^2\phi^2-\frac{1}{2}(Z_\chi-1)\partial^\mu\chi\partial_\mu\chi-\frac{1}{2}(Z_M-1)M^2\chi^2+Y\phi$$

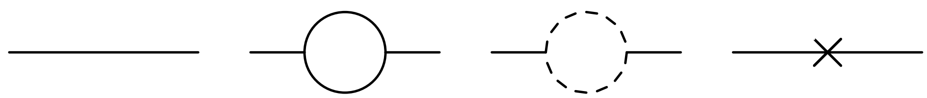
$\mathcal{L}_0$ represents the free fields, which we are not concerned with. The remaining lines give the interactions and vertex factors. Recall our rules for determining the vertex factors:

1. Replace all derivatives with ik, k positive for incoming particles
2. Add a factor of i
3. Erase the fields
4. Multiply through by the symmetry factor of the vertex

The vertices are therefore:

Vertex	Vertex Factor
	$iZ_g g\tilde{\mu}^{\varepsilon/2}$
	$iZ_h h\tilde{\mu}^{\varepsilon/2}$
	$-i\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]$
	$-i\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]$

Let's start by correcting the ϕ propagator. The ϕ propagator is given to one-loop by the following diagrams:



We need to calculate the self-energy of these diagrams. Recall that we assign:

- a factor of $-i$ overall (since the rules are generally presented for $i\Pi$)
- 1 to each external line
- $-i/(k^2 + m^2 - i\epsilon)$ for each internal line with momentum k
- the vertex factor (see table above) for each vertex
- $\int d^d\ell_i/(2\pi)^d$ for each loop
- $1/S$, where S is the product of any left-over symmetry factors from exchanges of internal propagators and vertices (those associates with external propagators affect all diagrams, and need not be treated separately here).

Note that the first diagram does not contribute to the self-energy, since it has no terms once the external lines are removed. Thus, these four diagrams give:

$$\Pi(k^2) = \frac{(-i)}{2} \int \frac{d^6\ell}{(2\pi)^6} \frac{(-i)}{\ell^2 + m^2 - i\epsilon} \frac{(-i)}{(k+\ell)^2 + m^2 - i\epsilon} (iZ_g g \tilde{\mu}^{\varepsilon/2})^2$$

$$+\frac{(-i)}{2} \int \frac{d^6\ell}{(2\pi)^6} \frac{(-i)}{\ell^2 + M^2 - i\epsilon} \frac{(-i)}{(k+\ell)^2 + M^2 - i\epsilon} (iZ_h h \tilde{\mu}^{\varepsilon/2})^2 + (-i)(-i) \left[(Z_\phi - 1)k^2 + (Z_m - 1)m^2\right]$$

We know that Z_g is of the form $1 + O(g^2)$. How do we know this? The whole point of the Zs is that they cancel the infinities in the loop corrections, and the lowest-order loop corrections as we have seen are always two orders above the tree-level diagram. We can therefore get rid of the Zs in the loop correction terms, since they will not contribute below O(g^4). Doing this and simplifying (note that no three-vertex terms can be drawn):

$$\Pi(k^2) = -\frac{ig^2\tilde{\mu}^\varepsilon}{128\pi^6} \int d^6\ell \frac{1}{\ell^2 + m^2 - i\epsilon} \frac{1}{(k+\ell)^2 + m^2 - i\epsilon}$$

$$-\frac{ih^2\tilde{\mu}^\varepsilon}{128\pi^6} \int d^6\ell \frac{1}{\ell^2 + M^2 - i\epsilon} \frac{1}{(k+\ell)^2 + M^2 - i\epsilon} - \left[(Z_\phi - 1)k^2 + (Z_m - 1)m^2\right] + O([g,h]^4)$$

Now we use Feynman's Formula to combine the denominators in both integrals. The result is equation 14.12:

$$\Pi(k^2) = -\frac{ig^2\tilde{\mu}^\varepsilon}{128\pi^6} \int d^6\ell \; dx \left[(\ell + xk)^2 + x(1-x)k^2 + m^2 - i\epsilon\right]^{-2}$$

$$-\frac{ih^2\tilde{\mu}^\varepsilon}{128\pi^6}\int d^6\ell\ dx\left[(\ell+xk)^2+x(1-x)k^2+M^2-i\epsilon\right]^{-2}-\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]+O([g,h]^4)$$

Now we define $q=\ell+xk$, so $d^q=d^d\ell$:

$$\Pi(k^2)=-\frac{ig^2\tilde{\mu}^\varepsilon}{128\pi^6}\int d^6q\ dx\left[q^2+x(1-x)k^2+m^2-i\epsilon\right]^{-2}$$

$$-\frac{ih^2\tilde{\mu}^\varepsilon}{128\pi^6}\int d^6q\ dx\left[q^2+x(1-x)k^2+M^2-i\epsilon\right]^{-2}-\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]+O([g,h]^4)$$

Next we make a Wick Rotation to kill these annoying ϵs.

$$\Pi(k^2)=\frac{g^2\tilde{\mu}^\varepsilon}{128\pi^6}\int d^6\overline{q}\ dx\left[\overline{q}^2+x(1-x)k^2+m^2\right]^{-2}$$

$$+\frac{h^2\tilde{\mu}^\varepsilon}{128\pi^6}\int d^6\overline{q}\ dx\left[\overline{q}^2+x(1-x)k^2+M^2\right]^{-2}-\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]+O([g,h]^4)$$

We solve the integrals over $\overline{q}$ by using equation 14.27, and remember that the last Feynman parameter is always integrated from 0 to 1:

$$\Pi(k^2)=-\frac{g^2\tilde{\mu}^\varepsilon}{128\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\int_0^1 dx\left[x(1-x)k^2+m^2\right]$$

$$-\frac{h^2\tilde{\mu}^\varepsilon}{128\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\int_0^1 dx\left[x(1-x)k^2+M^2\right]-\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]+O([g,h]^4)$$

Solving the remaining integral:

$$\Pi(k^2)=-\frac{g^2\tilde{\mu}^\varepsilon}{128\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\left[\frac{1}{6}k^2+m^2\right]$$

$$-\frac{h^2\tilde{\mu}^\varepsilon}{128\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\left[\frac{1}{6}k^2+M^2\right]-\left[(Z_\phi-1)k^2+(Z_m-1)m^2\right]+O([g,h]^4)$$

Now we choose the Z_ϕ that will cause the divergent $1/\varepsilon$ terms – and only those terms – to cancel (this is the $\overline{MS}$ renormalization scheme). The result is:

$$\boxed{Z_\phi=1-\frac{(h^2+g^2)\tilde{\mu}^\varepsilon}{6(4\pi)^3\varepsilon}+\dots}$$

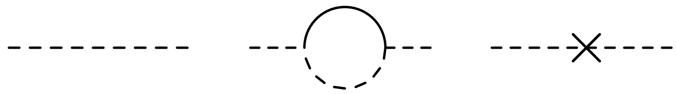
Similarly, we choose the Z_m:

$$\boxed{Z_m=1-\frac{(\frac{M^2}{m^2}h^2+g^2)\tilde{\mu}^\varepsilon}{(4\pi)^3\varepsilon}+\dots}$$

Not that it matters for our purposes, but let's also write the self-energy:

$$\Pi(k^2)=-\frac{g^2\tilde{\mu}^\varepsilon}{128\pi^3}(1-\gamma)\left[\frac{1}{6}k^2+m^2\right]-\frac{h^2\tilde{\mu}^\varepsilon}{128\pi^3}(1-\gamma)\left[\frac{1}{6}k^2+M^2\right]+\dots$$

Now let's correct the χ propagator, which is given to one-loop by the following diagrams:



The self-energy is given by:

$$\Pi(k^2) = (-i)\int \frac{d^6\ell}{(2\pi)^6}\frac{(-i)}{\ell^2+M^2-i\epsilon}\frac{(-i)}{(k+\ell)^2+m^2-i\epsilon}(iZ_h h\tilde{\mu}^{\varepsilon/2})^2$$
$$+(-i)(-i)\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]$$

Simplyifying this as before:

$$\Pi(k^2) = -\frac{ih^2\tilde{\mu}^\varepsilon}{64\pi^6}\int d^6\ell\frac{1}{\ell^2+M^2-i\epsilon}\frac{1}{(k+\ell)^2+m^2-i\epsilon}-\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]+O([g,h]^4)$$

Now we use Feynman's Formula to combine the denominators in the integral.

$$\Pi(k^2) = -\frac{ih^2\tilde{\mu}^\varepsilon}{64\pi^6}\int d^6\ell dx\left[(\ell+kx)^2+x(1-x)k^2+(m^2-M^2)x+M^2\right]^{-2}$$
$$-\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]+\dots$$

Now we switch to q, make a Wick Rotation (switching to $\overline{q}$), and integrate using equation 14.27:

$$\Pi(k^2) = -\frac{h^2\tilde{\mu}^\varepsilon}{64\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\int_0^1 dx\left[x(1-x)k^2+M^2+(m^2-M^2)x\right]-\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]+\dots$$

Now we do the remaining integral:

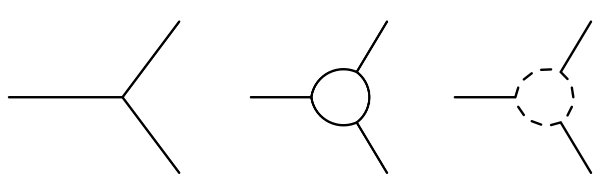
$$\Pi(k^2) = -\frac{h^2\tilde{\mu}^\varepsilon}{64\pi^3}\left(\frac{2}{\varepsilon}+1-\gamma\right)\left[\frac{1}{6}k^2+M^2+\frac{1}{2}(m^2-M^2)\right]-\left[(Z_\chi-1)k^2+(Z_M-1)M^2\right]+\dots$$

Choose Z_χ and Z_M to cancel the infinity only:

$$\boxed{Z_\chi = 1-\frac{h^2\tilde{\mu}^\varepsilon}{3(4\pi)^3\varepsilon}+\dots}$$

$$\boxed{Z_M = 1-\frac{h^2\tilde{\mu}^\varepsilon(1+\frac{m^2}{M^2})}{(4\pi)^3\varepsilon}+\dots}$$

Now let's calculate the ϕ^3 vertex. The diagrams are:



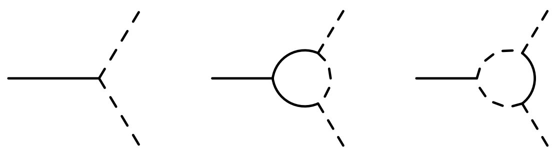
The second diagram is the same diagram that was considered in chapter 16. The third diagram is the same as the second, with $g \to h$ and $m \to M$. With these substitiutions, the answer can be obtained from equation 16.11:

$$V_3(k_1, k_2, k_3) = g\tilde{\mu}^{\varepsilon/2} + (Z_g - 1)g\tilde{\mu}^{\varepsilon/2} + \frac{g^3\tilde{\mu}^{3\varepsilon/2}}{(4\pi)^3}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) - \frac{1}{2}\int dF_3 \ln\left(\frac{D}{m^2}\right)\right]$$
$$+\frac{h^3\tilde{\mu}^{3\varepsilon/2}}{(4\pi)^3}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{M}\right) - \frac{1}{2}\int dF_3 \ln\left(\frac{D'}{M^2}\right)\right] + \dots \qquad (28.1.8)$$

where D is given by 16.5, and D' is the same as D with $m \to M$. Now we choose Z_g to cancel the infinity terms only:

$$\boxed{Z_g = 1 - \frac{(g^2 + h^3/g)\tilde{\mu}^{\varepsilon}}{(4\pi)^3\varepsilon} + \dots}$$

Examining this result, we see that Z_g depends only on the symmetry factors and the coupling constants; everything else (including the masses) is irrelevant. Finally we correct the $\phi\chi^2$ vertex. The diagrams are:



We compare these diagrams to the above, and make the necessary changes to equation (28.1.8). The symmetry factors are the same. The non-divergent terms in equation (28.1.8) are irrelevent. The vertex factors change to h, gh^2, and h^3, respectively. Thus,

$$V_3(k_1, k_2, k_3) = h\tilde{\mu}^{\varepsilon/2} + (Z_h - 1)h\tilde{\mu}^{\varepsilon/2} + \frac{gh^2\tilde{\mu}^{3\varepsilon/2}}{(4\pi)^3}\left[\frac{1}{\varepsilon} + \dots\right] + \frac{h^3\tilde{\mu}^{3\varepsilon/2}}{(4\pi)^3}\left[\frac{1}{\varepsilon} + \dots\right] + \dots$$

Thus,

$$\boxed{Z_h = 1 - \frac{(gh + h^2)\tilde{\mu}^{\varepsilon}}{(4\pi)^3\varepsilon} + \dots}$$

(b) The bare couplings are related to the renormalized ones via:

$$\boldsymbol{g_0 = Z_\phi^{-3/2} Z_g g \tilde{\mu}^{\varepsilon/2}}$$

$$\boldsymbol{h_0 = Z_\phi^{-1/2} Z_\chi^{-1} Z_h h \tilde{\mu}^{\varepsilon/2}}$$

Define:

$$\boldsymbol{G(g, h, \varepsilon) = \sum_{n=1}^{\infty} G_n(g, h)\varepsilon^{-n} = \ln\left(Z_\phi^{-3/2} Z_g\right)}$$

$$\boldsymbol{H(g, h, \varepsilon) = \sum_{n=1}^{\infty} H_n(g, h)\varepsilon^{-n} = \ln\left(Z_\phi^{-1/2} Z_\chi^{-1} Z_h\right)}$$

By requiring g_0 and h_0 to be independent of μ, and by assuming that $dg/d\mu$ and $dh/d\mu$ are finite as $\varepsilon \to 0$, show that

$$\boldsymbol{\mu \frac{dg}{d\mu} = -\frac{1}{2}\varepsilon g + \frac{1}{2} g \left(g \frac{\partial G_1}{\partial g} + h \frac{\partial G_1}{\partial h} \right)}$$

$$\boldsymbol{\mu \frac{dh}{d\mu} = -\frac{1}{2}\varepsilon h + \frac{1}{2} h \left(g \frac{\partial H_1}{\partial g} + h \frac{\partial H_1}{\partial h} \right)}$$

We have:

$$\ln g_0 = \ln \left(Z_\phi^{-3/2} Z_g \right) + \ln g + \frac{\varepsilon}{2} \ln \tilde{\mu}$$

Taking the derivative with respect to μ:

$$\frac{d \ln g_0}{d\mu} = \frac{\partial G}{\partial \mu} + \frac{1}{g}\frac{\partial g}{\partial \mu} + \frac{\varepsilon}{2\mu}$$

Since g_0 cannot depend on μ, this must equal zero. Thus:

$$\mu \frac{\partial g}{\partial \mu} + g\mu \frac{\partial G}{\partial \mu} = -\frac{\varepsilon g}{2}$$

Using the chain rule:

$$\mu \frac{\partial g}{\partial \mu} + g\mu \frac{\partial G}{\partial g}\frac{\partial g}{\partial \mu} + g\mu \frac{\partial G}{\partial h}\frac{\partial h}{\partial \mu} = -\frac{\varepsilon g}{2}$$

Repeating this analysis for h_0:

$$\mu \frac{\partial h}{\partial \mu} + h\mu \frac{\partial H}{\partial g}\frac{\partial g}{\partial \mu} + h\mu \frac{\partial H}{\partial h}\frac{\partial h}{\partial \mu} = -\frac{\varepsilon h}{2}$$

These two results can be easily expressed in a matrix:

$$\begin{pmatrix} 1 + g\frac{\partial G}{\partial g} & g\frac{\partial G}{\partial h} \\ h\frac{\partial H}{\partial h} & 1 + h\frac{\partial H}{\partial g} \end{pmatrix} \begin{pmatrix} \mu\frac{\partial g}{\partial \mu} \\ \mu\frac{\partial h}{\partial mu} \end{pmatrix} = -\frac{\varepsilon}{2} \begin{pmatrix} g \\ h \end{pmatrix}$$

This gives:

$$\begin{pmatrix} \mu\frac{\partial g}{\partial \mu} \\ \mu\frac{\partial h}{\partial mu} \end{pmatrix} = -\frac{\varepsilon}{2} \begin{pmatrix} 1 + g\frac{\partial G}{\partial g} & g\frac{\partial G}{\partial h} \\ h\frac{\partial H}{\partial h} & 1 + h\frac{\partial H}{\partial g} \end{pmatrix}^{-1} \begin{pmatrix} g \\ h \end{pmatrix}$$

Expanding around $g, h \approx 0$, the determinant of this matrix is one, so it is easy to take the inverse:

$$\begin{pmatrix} \mu\frac{\partial g}{\partial \mu} \\ \mu\frac{\partial h}{\partial mu} \end{pmatrix} = -\frac{\varepsilon}{2} \begin{pmatrix} 1 - h\frac{\partial H}{\partial g} & -g\frac{\partial G}{\partial h} \\ -h\frac{\partial H}{\partial h} & 1 - g\frac{\partial G}{\partial g} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}$$

This gives:

$$\begin{pmatrix} \mu\frac{\partial g}{\partial \mu} \\ \mu\frac{\partial h}{\partial mu} \end{pmatrix} = -\frac{\varepsilon}{2} \begin{pmatrix} g \\ h \end{pmatrix} + \frac{\varepsilon}{2} \begin{pmatrix} h\frac{\partial H}{\partial g} & g\frac{\partial G}{\partial h} \\ h\frac{\partial H}{\partial h} & g\frac{\partial G}{\partial g} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}$$

Expanding:

$$\begin{pmatrix} \mu\frac{\partial g}{\partial \mu} \\ \mu\frac{\partial h}{\partial mu} \end{pmatrix} = -\frac{\varepsilon}{2} \begin{pmatrix} g \\ h \end{pmatrix} + \frac{1}{2} \begin{pmatrix} h\frac{\partial H_1}{\partial g} & g\frac{\partial G_1}{\partial h} \\ h\frac{\partial H_1}{\partial h} & g\frac{\partial G_1}{\partial g} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} + \dots$$

The higher-order terms must be finite in the $\varepsilon \to 0$ limit. Thus:

$$\mu \frac{dg}{d\mu} = -\frac{1}{2}\varepsilon g + \frac{1}{2} g \left(g\frac{\partial G_1}{\partial g} + h\frac{\partial G_1}{\partial h} \right)$$

$$\mu \frac{dh}{d\mu} = -\frac{1}{2}\varepsilon h + \frac{1}{2} h \left(g\frac{\partial H_1}{\partial g} + h\frac{\partial H_1}{\partial h} \right)$$

as expected.

(c) Use your results from part (a) to compute the beta functions $\boldsymbol{\beta_g(g,h) = \lim_{\varepsilon\to 0} \mu\frac{dg}{d\mu}}$ and $\boldsymbol{\beta_h(g,h) = \lim_{\varepsilon\to 0} \mu\frac{dh}{d\mu}}$ We have:

$$G = \ln\left[Z_\phi^{-3/2} Z_g\right] = \ln\left[\left(1 - \frac{(g^2 + h^3/g)\tilde{\mu}^\varepsilon}{(4\pi)^3\varepsilon}\right)\left(1 - \frac{(h^2+g^2)\tilde{\mu}^\varepsilon}{6(4\pi)^3\varepsilon}\right)^{-3/2}\right]$$

This gives:

$$G = \ln\left[1 - \frac{g^2 + h^3/g}{(4\pi)^3\varepsilon} + \frac{(h^2+g^2)\tilde{\mu}^\varepsilon}{4(4\pi)^3\varepsilon} + \ldots\right]$$

Expanding, we find G_1:

$$G_1 = \frac{\tilde{\mu}^\varepsilon}{(4\pi)^3}\left(\frac{h^2}{4} - \frac{3g^2}{4} - h^3 g^{-1}\right)$$

Now we use the result from part (b):

$$\beta_g(g,h) = \lim_{\varepsilon\to 0}\left[-\frac{1}{2}\varepsilon g + \frac{1}{2} g^2 \frac{\partial G_1}{\partial g} + \frac{1}{2} gh \frac{\partial G_1}{\partial h}\right]$$

Thus:

$$\beta_g(g,h) = \frac{1}{2(4\pi)^3}\lim_{\varepsilon\to 0}\left[g^2\tilde{\mu}^\varepsilon\left(-\frac{3g}{2} + h^3 g^{-2}\right) + gh\tilde{\mu}^\varepsilon\left(\frac{h}{2} - 3h^2 g^{-1}\right)\right]$$

This gives:

$$\boxed{\beta_g(g,h) = \frac{1}{4(4\pi)^3}\left[-3g^3 + gh^2 - 4h^3\right]}$$

In the same way, we calculate the β function for h, I won't show all the calculus:

$$H_1 = \frac{1}{(4\pi)^3}\left[\frac{1}{12}g^2 - gh - \frac{7}{12}h^2\right]$$

$$\boxed{\beta_h = \frac{1}{(4\pi)^3}\left[\frac{g^2 h}{12} - gh^2 - \frac{7}{12}h^3\right]}$$

Note: See Srednicki's errata, this part of the problem may be stated incorrectly in your book. It it stated correctly here.

(d) Without loss of generality, we can choose g to be positive; h can then be positive or negative, and the difference is physically significant. (You should

understand why this is true.) For what numerical range(s) of h/g are β_g/g and β_h/h both negative? Why is this an interesting question?

See the Lagrangian: if the sign of g is changed, the Lagrangian can be invariant if the signs of ϕ and h are changed to compensate (h must be changed to keep Z_g the same, as found in part (a)). If g is set to positive, the sign of h is therefore constrained, and may be physically significant.

The question at hand is interesting because the β functions shows the coupling's dependence on energy. In the case at hand, an arbitrarily high energy means an arbitrarily weak coupling: this is asymptotic freedom.

As for the numerical range: we want to know when:

$$\beta_g = \frac{g^3}{4(4\pi)^3}\left[-3 + \frac{h^2}{g^2} - 4\frac{h^3}{g^3}\right] < 0$$

Solving this on Mathematica, we find that this is true when $h/g > -.083$.

Repeating this analysis for β_h, we find that the inequality is solved whenever $h/g < -1.7939$ or $h/g > .079634$.

Combining these, we find that the system is asymptotically free when $h/g > 0.079634$.