Srednicki Chapter 24

QFT Problems & Solutions

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Srednicki 24.1. Show that $\boldsymbol{\theta_{ij}}$ in equation 24.4 must be antisymmetric if R is orthogonal.

Orthogonality of R implies that:

$$I = RR^{-1} = RR^T$$

Writing this in index notation:

$$\delta_{ik} = R_{ij}(R^T)_{jk} = R_{ij}R_{kj}$$

Now use equation 24.4:

$$\delta_{ik} = (\delta_{ij} + \theta_{ij} + O(\theta^2))(\delta_{kj} + \theta_{kj} + O(\theta^2))$$

Expanding:

$$\delta_{ik} = \delta_{ij}\delta_{kj} + \delta_{ij}\theta_{kj} + \theta_{ij}\delta_{kj} + O(\theta^2)$$

Using the δs to eliminate j on the right hand side:

$$\delta_{ik} = \delta_{ki} + \theta_{ki} + \theta_{ik} + O(\theta^2)$$

$\delta_{ik} = \delta_{ki}$, so:

$$\delta_{ik} = \delta_{ik} + \theta_{ki} + \theta_{ik} + O(\theta^2)$$

Which of course gives, (to first order):

$$\theta_{ik} = -\theta_{ki}$$

as expected.

Srednicki 24.2. By considering the SO(N) transformation $\boldsymbol{R'^{-1}R^{-1}R'R}$, where $\boldsymbol{R}$ and $\boldsymbol{R'}$ are independent infinitesimal SO(N) transformations, prove equation 24.7.

Let's start by considering the transformation as instructed. Equation 24.4 gives:

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

Let's suppress the index notation to make this a little cleaner. Let's also work to second order (why? try it to first order, it will be trivial). Then:

$$R = 1 + \theta + \kappa\theta^2 + O(\theta^3)$$

where κ can be any real or imaginary number. Taylor expanding:

$$R^{-1} = 1 - \theta + (1-\kappa)\theta^2 + O(\theta^3)$$

Writing the series of infinitesimal transformations in question, we have:

$$R'^{-1}R^{-1}R'R = [1 - \theta' + (1-\kappa')\theta'^2][1 - \theta + (1-\kappa)\theta^2][1 + \theta' + \kappa'\theta'^2][1 + \theta + \kappa\theta^2] + O(\theta^3)$$

Now we expand this (remember that the θ terms do not necessarily commute!) and cancel the terms where possible, the result is that:

$$R'^{-1}R^{-1}R'R = 1 + \theta'\theta - \theta\theta' + O(\theta^3)$$

Now let's use equation 24.6. This equation is a little bit confusing because there are two θs, which will be identical when I drop the subscripts and superscripts. To avoid this, let's rename the right-hand side θ as $\bar{\theta}$. Then:

$$R'^{-1}R^{-1}R'R = 1 + (-i\bar{\theta}'T')(-i\bar{\theta}T) - (-i\bar{\theta}T)(-i\bar{\theta}'T') + O(\theta^3)$$

On the left-hand side, we have just a multiplication of four orthogonal, infinitesimal matrices, which is itself an infinitesimal matrix. On the right hand side, we simplify:

$$1 + \theta'' = 1 - \bar{\theta}'T'\bar{\theta}T + \bar{\theta}T\bar{\theta}'T' + O(\theta^3)$$

which is:

$$1 - i\bar{\theta}''T'' = 1 - \bar{\theta}'T'\bar{\theta}T + \bar{\theta}T\bar{\theta}'T' + O(\theta^3)$$

Recall that the $\bar{\theta}$ are just real parameters, so they commute with everything. Cancelling the delta terms, and ignoring higher-order terms, we have:

$$-i\bar{\theta}''T'' = -\bar{\theta}'\bar{\theta}T'T + \bar{\theta}\bar{\theta}'TT'$$

which is:

$$\bar{\theta}''T'' = -i\bar{\theta}'\bar{\theta}[T', T]$$

which gives:

$$\bar{\theta}'\bar{\theta}[T, T'] = -i\bar{\theta}''T''$$

Now if the $\bar{\theta}$ terms are zero, then the symmetry is Abelian, and equation 24.7 is trivial. If these terms are nonzero, then we can rewrite as:

$$[T, T'] = i\frac{-\bar{\theta}''}{\bar{\theta}'\bar{\theta}}T''$$

Changing notation, we have:

$$[T^a, T^b] = i\left(-\frac{\bar{\theta}^c}{\bar{\theta}^b\bar{\theta}^a}\right)T^c$$

Defining this term in parenthesis to be the structure constant, we have:

$$[T^a, T^b] = if^{abc}T^c$$

Note: I find Srednicki's solution to be a little misleading because he drops the second-order terms in equation 24.4. Hence, his solution is not fully general: he claims to work to second order, but restricts himself to the matrices which have no second-order terms. This was addressed above by the introduction of the second-order coefficient κ. Fortunately, κ cancels, so the result is the same.

Srednicki 24.3. (a) Find the Noether current $j^{\mu a}$ for the transformation of equation 24.6.

Srednicki 22.6 is:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\delta\phi(x)$$

Recall from chapter 22 that it is conventional to factor out the infinitesimal parameter. Hence,

$$\theta^a j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\delta\phi(x) \tag{24.3.1}$$

The first term of the Lagrangian can be rewritten as:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial_\nu\phi_i g^{\mu\nu}$$

Since g is a diagonal matrix:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i g^{\mu\mu}$$

Then:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = -\partial_\mu\phi_i g^{\mu\mu}$$

which is:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = -\partial^\mu\phi_i$$

Plugging this into equation (24.3.1) gives:

$$\theta^a j^{\mu a} = -\partial^\mu\phi_i \; \delta\phi(x) \tag{24.3.2}$$

Now, the transformation in question is:

$$\phi_i = R_{ij}\phi_j$$

which is:

$$\phi_i = (\delta_{ij} + \theta_{ij})\phi_j = \phi_i + \theta_{ij}\phi_j$$

From whence it follows that:

$$\delta\phi(x) = \theta_{ij}\phi_j$$

which gives, using equation (24.3.2):

$$\theta^a j^{\mu a} = -\partial^\mu \phi_i \theta_{ij} \phi_j$$

Using Srednicki 24.6:

$$\theta^a j^{\mu a} = -\partial^\mu \phi_i (-i) \theta^a (T^a)_{ij} \phi_j$$

which gives:

$$j^{\mu a} = i\partial^\mu \phi_i (T^a)_{ij} \phi_j$$

(b) Show that $[\phi_i, Q] = (T^a)_{ij}\phi_j$, where Q is the Noether charge.

$$[\phi_i, Q] = \int d^3y [\phi_i, j^0(y)]$$

$$\implies [\phi_i, Q] = \int d^3y [\phi_i(x), i\partial^0 \phi_i (T^a)_{ij} \phi_j(y)]$$

$$\implies [\phi_i, Q] = -i \int d^3y [\phi_i(x), \Pi_i](T^a)_{ij} \phi_j(y)$$

$$\implies [\phi_i, Q] = (T^a)_{ij} \phi_j$$

(c) Use this result, equation 24.7, and the Jacobi identity (see problem 2.8) to show that $[Q_A, Q_B] = if^{abc}Q_C$.

Recall that the Jacobi identity deals with commutators of the form $[[A, B], C]$. In this case, we'll consider the commutator $[[\phi_i, Q_a], Q_b]$. Then the Jacobi Identity states:

$$[[\phi_i, Q_a], Q_b] + [[Q_b, \phi_i], Q_a] = -[[Q_a, Q_b], \phi_i]$$

which implies:

$$[[\phi_i, Q_a], Q_b] - [[\phi_i, Q_b], Q_a] = -[[Q_a, Q_b], \phi_i]$$

Using the result from part (b):

$$[(T^a)_{ij}\phi_j, Q_b] - [(T^b)_{ij}\phi_j, Q_a] = -[[Q_a, Q_b], \phi_i]$$

Since we've used index notation, $(T^a)_{ij}$ is just a real parameter, and so:

$$(T^a)_{ij}[\phi_j, Q_b] - (T^b)_{ij}[\phi_j, Q_a] = -[[Q_a, Q_b], \phi_i]$$

Using the result from part (b) again:

$$(T^a)_{ij}(T^b)_{jk}\phi_k - (T^b)_{ij}(T^a)_{jk}\phi_k = -[[Q_a, Q_b], \phi_i]$$

Dropping the index notation, and labeling ϕ with the only nontrivial index (k and j will multiply out, but i is an "external" parameter, so we label ϕ with i), we have:

$$T^a T^b \phi_i - T^b T^a \phi_i = -[[Q_a, Q_b], \phi_i]$$

which is:

$$(T^a T^b - T^b T^a)\phi_i = -[[Q_a, Q_b], \phi_i]$$

implying:

$$[T^a, T^b]\phi_i = -[[Q_a, Q_b], \phi_i]$$

Using equation 24.7:

$$if^{abc}T^c\phi_i = -[[Q_a, Q_b], \phi_i]$$

which implies:

$$[\phi_i, [Q_a, Q_b]] = if^{abc}T^c\phi_i \tag{24.3.3}$$

We can also use the result of (b) directly:

$$[\phi_i, Q_c] = T^c\phi_i$$

Contracting both sides by if^{abc}:

$$[\phi_i, if^{abc}Q_c] = if^{abc}T^c\phi_i \tag{24.3.4}$$

where f^{abc} is a real parameter and commutes with everything; its placement in the above equation is merely suggestive. Of course, the right hand sides of equation (24.3.3) and equation (24.3.4) are equal. It follows that:

$$[Q^a, Q^b] = if^{abc}Q^c$$

as expected. [Note that I'm playing fast and loose with the position of the Latin characters: sometimes they are subscripts, sometimes superscripts. Since Latin characters represent spatial indices only, there is no significant difference between the two].

Srednicki 24.4. The elements of the group SO(N) can be defined as N $\times$ N matrices $\boldsymbol{R}$ that satisfy

$$\boldsymbol{R_{ii'}R_{jj'}\delta_{i'j'} = \delta_{ij}}$$

The elements of the *symplectic group* Sp(2N) can be defined as 2N $\times$ 2N matrices $\boldsymbol{S}$ that satisfy

$$\boldsymbol{S_{ii'}S_{jj'}\eta_{i'j'} = \eta_{ij}}$$

where the *symplectic metric* $\boldsymbol{\eta_{ij}}$ is antisymmetric, $\boldsymbol{\eta_{ij} = -\eta_{ji}}$, and squares to minus the identity: $\boldsymbol{\eta^2 = -I}$. One way to write $\boldsymbol{\eta}$ is

$$\boldsymbol{\eta} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix}$$

where I is the N $\times$ N identity matrix. Find the number of generators of Sp(2N).

Recall that the generator is the first-order term in the Taylor Expansion of the group being imposed infinitesimally. Taking this as an infinitesimal transformation, let's write:

$$S = 1 + \theta$$

where θ is the generator (after factoring out i and the differential). We can write:

$$\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{24.3.5}$$

Now let's examine the condition for the group. We have:

$$S_{ii'}S_{jj'}\eta_{i'j'} = \eta_{ij}$$

We can rewrite this as:

$$S_{ii'}\eta_{i'j'}(S^T)_{j'j} = \eta_{ij}$$

Dropping the index notation:

$$S\eta S^T = \eta$$

This gives:

$$(1+\theta)\eta(1+\theta^T) = \eta$$
$$\implies \eta + \eta\theta^T + \theta\eta + \theta\eta\theta^T = \eta$$
$$\implies \eta\theta^T + \theta\eta + \theta\eta\theta^T = 0$$

The last term can be dropped since it contains two differentials. Hence,

$$\eta\theta^T + \theta\eta = 0$$

Now we have θ given by (24.3.5) and η given by (in one representation) equation 24.16. Then:

$$0 = \begin{pmatrix} B^T & D^T \\ -A^T & -C^T \end{pmatrix} + \begin{pmatrix} -B & A \\ -D & C \end{pmatrix} = \begin{pmatrix} -(B-B^T) & A+D^T \\ -(A+D^T) & C-C^T \end{pmatrix}$$

The number of generators is given by the number of degrees of freedom of θ – a maximum of $4N^2$. A has no restrictions, and contributes N^2 degrees of freedom. B and C have to equal their own transposes, so they have $\frac{1}{2}(N)(N+1)$ degrees of freedom each. D must be the negative transpose of A, so it contributes no degrees of freedom. Combining these, we find that there are $\boxed{2N^2+N}$ generators. This will be independent of the form of the generator, so our use of equation 24.16 is valid.