Srednicki 22.1. For the Noether current of equation 22.6 and assuming that $\delta \phi$ does not involve time derivatives, use the canonical commutation relations to show that $[\phi, Q] = i\delta \phi$.

Using equation 22.17:

$$[\phi, Q] = [\phi(x), \int d^3x j^0(x')]$$

Using equation 22.6:

$$[\phi, Q] = \int d^3x [\phi(x), -\frac{\partial L(x')}{\partial (\partial_0 \phi_b(x'))}\delta \phi_b(x')]$$

Using the definition of the conjugate momentum:

$$[\phi, Q] = \int d^3x [\phi(x), \Pi_b(x')\delta \phi_b(x')]$$

Since $\delta \phi$ doesn’t have any time derivatives, we expect that it will commute with $\phi$. Then:

$$[\phi, Q] = \int d^3x [\phi(x), \Pi_b(x')]\delta \phi_b(x')$$

Now we use equation 3.28.

$$[\phi, Q] = \int d^3x i\delta^3(x - x')\delta_{ab}\delta \phi_b(x')$$

which gives:

$$[\phi, Q] = i\delta \phi$$

as expected. Two notes, both about equation 3.28:

- We assume that these are both taken at equal times, i.e. that $x^0 = x'^0$. This is fine because $Q$ should be constant in time, per the discussion in the text.

- We assume that the indices $a$ and $b$ are equal. If not, then we have two different fields, which of course commute.
Srednicki 22.2. Use the canonical commutation relations to verify equation 22.38.

Using equation 22.35:

\[ [\phi(x), P^\mu] = \int d^3x' [\phi(x), T^{0\mu}(x')] \]  

(22.2.1)

Using equation 22.33:

\[ [\phi(x), P^j] = -\int d^3x' [\phi(x), \Pi(x')\nabla^j \phi(x')] \]

We can take the \( \nabla^j \phi(x') \) out of the commutator, since it obviously commutes with \( \phi(x) \). Then,

\[ [\phi(x), P^j] = -\int d^3x' [\phi(x), \Pi(x')] \nabla^j \phi(x') \]

Now we use the commutation relations.

\[ [\phi(x), P^j] = -i \int d^3x' \delta^3(x-x') \nabla^j \phi(x') \]

And we do the integral:

\[ [\phi(x), P^j] = \frac{1}{i} \nabla^j \phi(x) \]  

(22.2.2)

Now we go back to equation (22.2.1), this time choosing to insert 22.32:

\[ [\phi(x), P^0] = \int d^3x' [\phi(x), \frac{1}{2} \Pi^2(x')+\frac{1}{2}(\nabla \phi)^2+V(\phi)] \]

Note that we have implicitly set the right side to have a subscripts, since any other subscript refers to a different particle, and therefore commutes. Now all these terms on the right hand side commute with \( \phi \) except the second one. So,

\[ [\phi(x), P^0] = \frac{1}{2} \int d^3x' [\phi(x), \Pi^2(x')] \]

which gives, using the properties of commutators:

\[ [\phi(x), P^0] = \frac{1}{2} \int d^3x' [\phi(x), \Pi(x')]\Pi(x') + \frac{1}{2} \int d^3x' \Pi(x')[\phi(x), \Pi(x')] \]

From which it follows:

\[ [\phi(x), P^0] = \int d^3x' i\delta^3(x-x')\Pi(x') \]

Doing the integral:

\[ [\phi(x), P^0] = -\frac{1}{i} \Pi(x) \]

Now remember that \( \Pi \) is the time derivative of \( \phi \), so:

\[ [\phi(x), P^0] = -\frac{1}{i} \frac{d}{dt} \phi(x) \]  

(22.2.3)
Combining (22.2.3) and (22.2.2) into one equation, we have:

\[ \lbrack \phi(x), P^{0} \rbrack = \frac{1}{i} \partial^{\mu} \phi(x) \]

as expected.

**Srednicki 22.3.** (a) With \( T^{\mu\nu} \) given by equation 22.31, compute the equal-time commutators \( [T^{00}(x), T^{00}(y)], [T^{0i}(x), T^{00}(y)], [T^{0i}(x), T^{0j}(y)] \).

This is a very long problem, so I want to note from the beginning that I will set \( \phi_a = \phi \), since all commutators vanish if they compare two different scalar fields. I also write the dependent variables only when there is a potential ambiguity. Equation 22.31 gives:

\[ [T^{00}(x), T^{00}(y)] = \left[ \frac{1}{2} \Pi^2(x) + \frac{1}{2} (\nabla \phi(x))^2 + V(\phi(x)), \frac{1}{2} \Pi^2(y) + \frac{1}{2} (\nabla \phi(y))^2 + V(\phi(y)) \right] \]

Note that we have \( (\nabla \phi)^2 = \nabla^j \phi \nabla_j \phi \). There are two different \( (\nabla \phi)^2 \) terms (one on each side), so we should write both of them out, using different indices to represent summation. But note that the terms where the two \( \nabla \) terms meet will all vanish (by equation 2.38), so it is simplest to just set \( \nabla^i = \nabla^j = \nabla \); all the \( \nabla s \) will have the same index in the end.

\[ [T^{00}(x), T^{00}(y)] = \frac{1}{4} [\Pi^2(x), (\nabla \phi(y))^2] + \frac{1}{4} [\Pi^2(x), V(\phi(y))] - \frac{1}{4} [\Pi^2(y), (\nabla \phi(x))^2] - \frac{1}{4} [\Pi^2(y), V(\phi(x))] \]

where we have neglected the terms which vanish (recalling that the canonical commutation relations require that \( \Pi \) and \( \phi \) commute with themselves, even if the variables are different). Next we use the properties of the commutators:

\[ [T^{00}(x), T^{00}(y)] = \frac{1}{4} [\Pi^2(x), \nabla \phi(y)] \nabla \phi(y) + \frac{1}{4} \nabla \phi(y) \nabla \phi(y) + \frac{1}{2} [\Pi^2(x), V(\phi(y))] \]

\[ -\frac{1}{4} [\Pi^2(y), \nabla \phi(x)] \nabla \phi(x) - \frac{1}{4} \nabla \phi(x) \nabla \phi(x) - \frac{1}{2} [\Pi^2(y), V(\phi(x))] \]

Most of these terms have a nabla (spatial derivative). This can be moved outside the integral, since the spatial variable in quantum field theory is just a label, not an operator. Then:

\[ [T^{00}(x), T^{00}(y)] = \frac{1}{4} \nabla_y [\Pi^2(x), \phi(y)] \nabla \phi(y) + \frac{1}{4} \nabla_y \phi(y) \nabla [\Pi^2(x), \phi(y)] + \frac{1}{2} [\Pi^2(x), V(\phi(y))] \]

\[ -\frac{1}{4} \nabla_x [\Pi^2(y), \phi(x)] \nabla \phi(x) - \frac{1}{4} \nabla \phi(x) \nabla_x [\Pi^2(y), \phi(x)] - \frac{1}{2} [\Pi^2(y), V(\phi(x))] \]

We can easily show that \( [\Pi^2(x), \phi(y)] = -2i \delta^3(x-y) \Pi(x) \) [we also showed this in problem 22.2]. Then:

\[ [T^{00}(x), T^{00}(y)] = \frac{1}{2i} \nabla_y \delta^3(x-y) \Pi(x) \nabla \phi(y) + \frac{1}{2i} \nabla \phi(y) \nabla_y \delta^3(x-y) \Pi(x) + \frac{1}{2} [\Pi^2(x), V(\phi(y))] \]

\[ -\frac{1}{2i} \nabla_x \delta^3(x-y) \Pi(y) \nabla \phi(x) - \frac{1}{2i} \nabla \phi(x) \nabla_x \delta^3(x-y) \Pi(y) - \frac{1}{2} [\Pi^2(y), V(\phi(x))] \]
Now, what to do with \( V(\phi(x)) \)? Whatever it is, let’s do a Taylor expansion, so we have

\[
[\Pi^2(y), V(\phi(x))] = \sum_n [\Pi^2(y), c_n \phi(x)^n]
\]

Now our properties of commutators give us

\[
\sum_n nc_n \phi(x)^{i-1} [\Pi^2(y), \phi(x)]
\]

from which we have, remembering that \( \phi \) always commutes with itself,

\[
\frac{1}{i} \sum_n nc_n \phi(x)^{i-1} \delta^3(x - y) \Pi(x) + \frac{1}{i} \sum_n nc_n \Pi(x) \phi(x)^{i-1} \delta^3(x - y)
\]

Notice that we can write this as

\[
\frac{1}{i} \Pi(x) \frac{\delta V(\phi(x))}{\delta \phi} \delta^3(x - y) + \frac{1}{i} \frac{\delta V(\phi(x))}{\delta \phi} \delta^3(x - y) \Pi(x)
\]

By the way, this highly useful derivation is absent from most lists of identities, presumably because our assumption that all functions can be written as a Taylor Series is questionable (but it’s OK for our purposes, our potentials should be polynomial functions already, certainly nothing more complicated than an exponential). Using this in our expression for \([T^{00}(x), T^{00}(y)]\), we have:

\[
[T^{00}(x), T^{00}(y)] = \frac{1}{2i} \nabla_y \delta^3(x - y) \Pi(x) \nabla \phi(y) + \frac{1}{2i} \nabla \phi(y) \nabla_y \delta^3(x - y) \Pi(x) + \frac{1}{2i} \frac{\delta V}{\delta \phi} \delta^3(x - y) \Pi(x)
\]

\[
+ \frac{1}{2i} \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x - y) - \frac{1}{2i} \nabla_x \delta^3(x - y) \Pi(y) \nabla \phi(x) - \frac{1}{2i} \nabla \phi(x) \nabla_x \delta^3(x - y) \Pi(y) - \frac{1}{2i} \frac{\delta V}{\delta \phi} \delta^3(x - y) \Pi(y)
\]

\[
- \frac{1}{2i} \Pi(y) \frac{\delta V}{\delta \phi} \delta^3(x - y)
\]

Finally, we rewrite this a bit, and reinsert our nabla product index for clarity:

\[
[T^{00}(x), T^{00}(y)] = -\frac{i}{2} \Pi(x) \left( \nabla^j \phi(y) \nabla^j + \frac{\delta V}{\delta \phi} \right) \delta^3(x - y) + \frac{i}{2} \Pi(y) \left( \nabla^j \phi(x) \nabla^j + \frac{\delta V}{\delta \phi} \right) \delta^3(x - y)
\]

\[
- \frac{i}{2} \left( \nabla^j \phi(y) \nabla^j + \frac{\delta V}{\delta \phi} \right) \delta^3(x - y) \Pi(x) + \frac{i}{2} \left( \nabla^j \phi(x) \nabla^j + \frac{\delta V}{\delta \phi} \right) \delta^3(x - y) \Pi(y)
\]

Next:

\[
[T^{00}(x), T^{0i}(y)] = \left[ \frac{1}{2} \Pi^2(x) + \frac{1}{2} \nabla \phi(x) + V(\phi), -\Pi(y) \nabla \phi \right]
\]

As before, we use the commutation rules to separate this, using \( j \) superscripts to indicate multiplication where needed. We achieve:

\[
[T^{00}(x), T^{0i}(y)] = \frac{1}{2} \Pi(y) \Pi^j(x) \nabla^j \left[ \phi(y), \Pi^j(x) \right] + \frac{1}{2} \Pi(y) \nabla^j \left[ \phi(y), \Pi^j(x) \right] \Pi^i(x)
\]

\[
- \frac{1}{2} \nabla_x \left[ \phi(x), \Pi, y \right] \nabla^j \phi(x) \nabla^i \phi(y) - \frac{1}{2} \nabla^j \phi(x) \nabla^i \left[ \phi(x), \Pi(y) \right] \nabla^i \phi(y)
\]
\[ + \Pi(y) \nabla^i [\phi(y), V(\phi)] + [\Pi(y), V(\phi)] \nabla^i \phi(y) \]

The first term on the third line vanishes because \( \phi \) always commutes with itself. The last term is deconstructed into a Taylor series as before. The first four terms can be calculated directly. The result is:

\[ [T^{00}(x), T^{0i}(y)] = i \frac{1}{2}\Pi(y)\Pi(x) \nabla^i \delta^3(x - y) + i \frac{1}{2}\Pi(y) \{ \nabla^i \delta^3(x - y) \} \Pi(x) \]

\[-i \left\{ \nabla^j \delta^3(x - y) \right\} \nabla^j \phi(x) \nabla^i \phi(y) - \frac{i}{2} \nabla^j \delta^3(x - y) \} \nabla^i \phi(y) - i \frac{\delta V}{\delta \phi} \nabla^i \phi(y) \delta^3(x - y) \]

Note that a few of the \( j \) superscripts (indicating multiplication) disappeared, since multiplying any operator by the delta function is assumed. Cleaning this up, we have:

\[ [T^{00}(x), T^{0i}(y)] = -i \left[ \nabla^i \phi(x) \nabla^j \phi(y) + \frac{\delta V}{\delta \phi} \right] \delta^3(x - y) + i \Pi(y) \Pi(x) \nabla^i \phi(y) \]

Finally:

\[ [T^{0i}, T^{0j}(y)] = [\Pi(x) \nabla^i \phi(x), \Pi(y) \nabla^j \phi(y)] \]

Using the commutator rules:

\[ [T^{0i}, T^{0j}(y)] = \Pi(x) \{ \nabla^i [\phi(x), \Pi(y)] \} \nabla^j \phi(y) + \Pi(y) \{ \nabla^j [\Pi(x), \phi(y)] \} \nabla^i \phi(x) \]

Now we use the commutation relations (equation 3.28):

\[ [T^{0i}, T^{0j}(y)] = i\Pi(x) \nabla^i \delta^3(x - y) \nabla^j \phi(y) - i \Pi(y) \nabla^j \delta^3(x - y) \nabla^i \phi(x) \]

(b) Use your results to verify equations 2.17, 2.19, and 2.20.

We now have to verify nine equations. Let’s start with this one:

\[ [P_i, H] = \left[ \int d^3x T^{0i}(x), \int d^3y \Pi^2(y) + \frac{1}{2}(\nabla \phi(y))^2 + V(\phi(y)) \right] \]

which gives:

\[ [P_i, H] = \int d^3x d^3y \left[ T^{0i}(x), T^{00}(y) \right] \]

We worked out this commutator in part (a), the result is:

\[ [P_i, H] = \int d^3x d^3y \left\{ i \left[ \nabla^i \phi(x) \nabla^j \phi(y) + \frac{\delta V}{\delta \phi} \right] \delta^3(x - y) \nabla^j \phi(y) - i \Pi(y) \Pi(x) \nabla^i \phi(y) \right\} \]

Breaking this up and doing some integration by parts (remember that integration by parts introduces a negative sign and shifts, for example, \( \nabla_x \) from its current location to all other functions of \( x \)):

\[ [P_i, H] = i \int d^3x d^3y \left[ - \nabla^i \nabla^j \phi(x) \delta^3(x - y) \nabla^j \phi(y) \right] + i \frac{\delta V}{\delta \phi} \nabla^i \phi(y) + i \int d^3x d^3y \left( \nabla^i \Pi(y) \right) \Pi(x) \delta^3(x - y) \]
Using the delta functions, we have:

$$[P_i, H] = -i \int d^3 y \left[ \nabla^2 \phi(y) \nabla^i \phi(y) - i \nabla^i \Pi(y) \Pi(y) - \frac{\delta V}{\delta \phi} \nabla^i \phi(y) \right]$$

Now, note that we can rewrite this:

$$[P_i, H] = i \int d^3 y \nabla^i \left[ \frac{1}{2} \nabla^j \phi \nabla^j \phi + \frac{1}{2} \Pi(y) \Pi(y) + V(\phi) \right]$$

[If this first term does not seem obvious, use the product rule, then reverse the order of the derivatives (they commute) and integrate by parts.] Now remember that the integral of a total derivative vanishes assuming suitable boundary conditions at infinity (by the chain rule). So,

$$[P_i, H] = 0$$

as expected.

Second:

$$[P_i, P_j] = \int d^3 x d^3 y \left[ T^{0i}(x), T^{0j}(y) \right]$$

Using our result from part (a), we have:

$$[P_i, P_j] = \int d^3 x d^3 y \left[ i \Pi(x) \nabla^i \phi(y) \nabla^j \phi(y) - i \Pi(y) \nabla^i \phi(x) \nabla^j \phi(x) \right]$$

Integrating by parts:

$$[P_i, P_j] = i \int d^3 x d^3 y \left[ -\nabla^i \Pi \nabla^j \phi \delta^3(x - y) + \nabla^j \Pi \nabla^i \phi \delta^3(x - y) \right]$$

Using the delta functions:

$$[P_i, P_j] = i \int d^3 x \left[ -\nabla^i \Pi \nabla^j \phi + \nabla^j \Pi \nabla^i \phi \right]$$

Now we can integrate by parts on the last term, commute the two derivatives, then integrate by parts again. This causes the two terms to cancel, giving:

$$[P_i, P_j] = 0$$

This completes our verification of equation 2.20.

Next, we’ll calculate:

$$[J_i, H] = \int d^3 y \left[ \frac{1}{2} \varepsilon_{ijk} M^{jk}, T^{00}(y) \right]$$

$$\implies [J_i, H] = \frac{1}{2} \varepsilon_{ijk} \int d^3 x d^3 y \left[ M^{0jk}(x), T^{00}(y) \right]$$
\[ [J_i, H] = \frac{1}{2} \varepsilon_{ijk} \int d^3x \varepsilon^j \left[ x^j T^{0k}(x) - x^k T^{0j}(x) \right] \]
\[ [J_i, H] = \frac{1}{2} \varepsilon_{ijk} \int d^3x \varepsilon^j \left[ T^{0k}(x), T^{00}(y) \right] + \frac{1}{2} \varepsilon_{ikj} \int d^3x \varepsilon^j \left[ T^{0j}(x), T^{00}(y) \right] \]

Does this integral seem familiar? It’s exactly what we got when we calculated \([P_i, H]\), only there’s an additional term of \(x^j\) (shift the indices needed for the different terms in the expression). When we do the integration by parts, we will have something equivalent to \(\nabla_x^j (x^k \nabla_j \phi(x))\). But, notice that our levi-civita symbol \((\varepsilon_{ijk})\) requires that \(j\) and \(k\) be different variables. The result is that the \(x^k\) does not affect anything (it is a constant with respect to the derivative), and ends up as another term inside the constant derivative:

\[ [J_i, H] = \frac{i}{2} \varepsilon_{ijk} \int d^3y \nabla^i \left[ y^j \frac{1}{2} \nabla_k \phi \nabla^k \phi + y^j \frac{1}{2} \Pi(y) \Pi(y) + y^j \nabla \phi \right] + (j \leftrightarrow k) \]

which, as before, gives zero:

\[ [J_i, H] = 0 \]

Next:

\[ [J_i, P_j] = \frac{1}{2} \varepsilon_{i\ell k} \mathcal{M}^{\ell k} \int d^3y T^{0j}(y) \]

which gives:

\[ [J_i, P_j] = \frac{1}{2} \varepsilon_{i\ell k} \int d^3x \varepsilon^j \left[ \mathcal{M}^{0\ell k}(x, y) \int T^{0j}(y) \right] \]
\[ [J_i, P_j] = \frac{1}{2} \varepsilon_{i\ell k} \int d^3x \varepsilon^j \left[ x^\ell T^{0k}(x) - x^k T^{0\ell}(x), \int T^{0j}(y) \right] \]
\[ [J_i, P_j] = \frac{1}{2} \varepsilon_{i\ell k} \int d^3x \varepsilon^j \left[ T^{0k}(x), \int T^{0j}(y) \right] + (k \leftrightarrow \ell) \]

Note that the sign is positive due to the Levi-Cevita symbol. Using our result from part (a), we have:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i\ell k} \int d^3x \varepsilon^j \left\{ \Pi(x) \nabla^j_5 \delta^\ell (x - y) \nabla^j \phi(y) - \Pi(y) \nabla^j_5 \delta^\ell (x - y) \nabla^j \phi(x) + (k \leftrightarrow \ell) \right\} \]

Integrating by parts:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i\ell k} \int d^3x \varepsilon^j \left\{ x^\ell \nabla^j_5 \Pi(y) \nabla^j \phi(x) \delta^\ell (x - y) - \nabla^j \left( x^\ell \Pi(x) \right) \nabla^j \phi(y) \delta^\ell (x - y) + (k \leftrightarrow \ell) \right\} \]

Using the delta function:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i\ell k} \int d^3x \left[ x^\ell \nabla^j_5 \Pi \nabla^j \phi - \nabla^j \left( x^\ell \Pi \right) \nabla^j \phi + (k \leftrightarrow \ell) \right] \]

Evaluating the derivative in the second term, we have:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i\ell k} \int d^3x \left[ x^\ell \nabla^j_5 \Pi \nabla^j \phi - x^\ell \nabla^j \Pi \nabla^j \phi - \delta^{ij} \nabla^j \phi + (k \leftrightarrow \ell) \right] \]
Now we use integration by parts in the second term:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i \ell k} \int d^3 x \left[ x^\ell \nabla^j \Pi \nabla^k \phi + \nabla^k (x^\ell \nabla^j \phi) \Pi - \delta^{k \ell} \Pi \nabla^j \phi + (k \leftrightarrow \ell) \right] \]

When evaluating this derivative with the product rule, one of the terms cancels with the last term. So:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i \ell k} \int d^3 x \left[ x^\ell \nabla^j \Pi \nabla^k \phi + x^\ell \nabla^k (\nabla^j \phi) \Pi + (k \leftrightarrow \ell) \right] \]

Reversing the order of the two derivatives:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i \ell k} \int d^3 x \left[ x^\ell \nabla^j \Pi \nabla^k \phi + x^\ell \nabla^j (\nabla^k \phi) \Pi + (k \leftrightarrow \ell) \right] \]

Integrating by parts again:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i \ell k} \int d^3 x \left[ x^\ell \nabla^j \Pi \nabla^k \phi - \nabla^j (x^\ell \Pi) \nabla^k \phi + (k \leftrightarrow \ell) \right] \]

Doing the derivative this time, only one term remains:

\[ [J_i, P_j] = \frac{i}{2} \varepsilon_{i \ell k} \int d^3 x \left[ -\Pi \delta^{j \ell} \nabla^k \phi + (k \leftrightarrow \ell) \right] \]

which gives (remembering that the plus sign indicates that the second term should have the same sign as the first term):

\[ [J_i, P_j] = \frac{i}{2} \int d^3 x \left[ -\varepsilon_{i \ell k} \Pi \delta^{j \ell} \nabla^k \phi - \varepsilon_{i \ell k} \Pi \delta^{jk} \nabla^\ell \phi \right] \]

Using the Levi-Cevita symbol:

\[ [J_i, P_j] = \frac{i}{2} \int d^3 x \left[ -\varepsilon_{i \ell k} \Pi \delta^{j \ell} \nabla^k \phi - \varepsilon_{i \ell k} \Pi \delta^{jk} \nabla^\ell \phi \right] \]

Using the delta function:

\[ [J_i, P_j] = \frac{i}{2} \int d^3 x \left[ -\varepsilon_{ijkl} \Pi \nabla^k \phi - \varepsilon_{ijkl} \Pi \nabla^l \phi \right] \]

Notice that \( \ell \) is now just a dummy variable, just like \( k \). Let’s redefine for consistency \( \ell \rightarrow k \). Then,

\[ [J_i, P_j] = -i \varepsilon_{ij k} \int d^3 x \Pi \nabla^k \phi \]

which gives:

\[ [J_i, P_j] = i \varepsilon_{ij k} P^k \]

Then:

\[ [K_i, H] = [M^{i0}, H] \]

\[ \implies [K_i, H] = \left[ \int d^3 x M^{i0}(x), \int d^3 y T^{00}(y) \right] \]
\[ [K_i, H] = \int d^3x d^3y \left[ x^i T^{00}(x) - x^0 T^{0i}(x), T^{00}(y) \right] \]

Note that \( x^0 = t \). Let’s choose to evaluate the commutator at \( t = 0 \). Since commutators should not be time-dependent, the value at \( t = 0 \) should equal the commutator at all other times. Hence,

\[ [K_i, H] = \int d^3x d^3y x^i \left[ T^{00}(x), T^{00}(y) \right] \]

Using our result from part (a):

\[ [K_i, H] = \frac{i}{2} \int d^3x d^3y \left\{ -x^i \Pi(x) \nabla^j \phi(y) \nabla_y^j \delta^3(x - y) - x^i \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x - y) + \Pi(y) x^i \nabla^j \phi(x) \nabla_y^j \delta^3(x - y) \right\} \]

\[ + \Pi(y) x^i \frac{\delta V}{\delta \phi} \delta^3(x - y) - \nabla^j \phi(y) \nabla_y^j \delta^3(x - y) \Pi(x) x^i - \frac{\delta V}{\delta \phi} \delta^3(x - y) \Pi(x) x^i + x^i \nabla^j \phi(x) \nabla_y^j \delta^3(x - y) \Pi(y) \]

\[ + \frac{\delta V}{\delta \phi} \delta^3(x - y) \Pi(y) x^i \right\} \]

Splitting up this integral, and integrating by parts as necessary:

\[ [K_i, H] = \frac{i}{2} \left\{ \int d^3x d^3y \left( \nabla^2 \phi(y) \Pi(x) x^i \delta^3(x - y) \right) + \int d^3x d^3y \left( -x^i \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x - y) \right) \right\} \]

\[ + \int d^3x d^3y \left( -\Pi(y) \nabla_x^j \left[ x^i \nabla^j \phi(x) \right] \right) \delta^3(x - y) + \int d^3x d^3y \left( \Pi(y) x^i \frac{\delta V}{\delta \phi} \delta^3(x - y) \right) \]

\[ + \int d^3x d^3y \left( \nabla^2 \phi(y) \Pi(x) x^i \delta^3(x - y) \right) + \int d^3x d^3y \left( \Pi(x) x^i \frac{\delta V}{\delta \phi} \delta^3(x - y) \right) \]

\[ + \int d^3x d^3y \left( -\delta_x^j \left[ x^i \nabla^j \phi(x) \right] \right) \Pi(y) \delta^3(x - y) + \int d^3x d^3y \left( x^i \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x - y) \right) \}

Using the delta functions, and suppressing the variable (x), which is now the only variable, we have:

\[ [K_i, H] = \frac{i}{2} \left\{ \int d^3x \nabla^2 \phi \Pi x - \int d^3x x^i \Pi \frac{\delta V}{\delta \phi} - \int d^3x \Pi \left( x^i \nabla^2 \phi + \nabla^j \phi \delta_{ij} \right) + \int d^3x \Pi x^i \frac{\delta V}{\delta \phi} \right\} \]

\[ \int d^3x \nabla^2 \phi \Pi x^i - \int d^3x x^i \Pi \frac{\delta V}{\delta \phi} x^i - \int d^3x \Pi \left( x^i \nabla^2 \phi + \delta_{ij} \nabla^j \phi \right) + \int d^3x x^i \Pi \frac{\delta V}{\delta \phi} \}

The second, fourth, sixth, and eighth terms cancel. Two of the remaining terms combine. We’re left with:

\[ [K_i, H] = i \int d^3x \nabla^j (\nabla^i \phi) \Pi x^i - \nabla^j (x^i \nabla^j \phi) \Pi \]

(22.3.1)

The first and fifth terms cancel with the first term in the third and seventh terms. The two remaining terms combine to give:

\[ [K_i, H] = -i \int d^3x \Pi \delta^ij \nabla^j \phi \]
\[ [K_i, H] = -i \int d^3 x \Pi^i \phi \]

\[ [K_i, H] = i \int d^3 x T^{0i}(x) \]

where we reinserted the variable \((x)\), giving:

\[ [K_i, H] = i P^i \]

Next:

\[ [K_i, P_j] = \left[ M^{i0}, \int d^3 y T^{0j}(y) \right] \]

\[ [K_i, P_j] = \left[ \int d^3 x M^{0i}(x), \int d^3 y T^{0j}(y) \right] \]

\[ [K_i, P_j] = \int d^3 x d^3 y \left[ x^i T^{00}(x) - x^0 T^{0i}(x), T^{0j}(y) \right] \]

Note that \(x^0 = t\). Let’s choose to evaluate the commutator at \(t = 0\). Since commutators should not be time-dependent, the value at \(t = 0\) should equal the commutator at all other times. Hence,

\[ [K_i, P_j] = \int d^3 x d^3 y x^i \left[ T^{00}(x), T^{0j}(y) \right] \]

Using our results from part a, we have:

\[ [K_i, P_j] = -i \left\{ \int d^3 x d^3 y x^i \nabla^k_x \phi(x) \nabla^k_x \delta^3(x - y) \nabla^j_y \phi(y) + \int d^3 x d^3 y \frac{\delta V}{\delta \phi} \delta^3(x - y) \nabla^j_y \phi(y) \right\} \]

Now we integrate by parts in the first and third terms, and use the delta function in the second term. The result is:

\[ [K_i, P_j] = -i \left\{ -\int d^3 x d^3 y \nabla^k_x \phi(x) \nabla^j_y \phi(y) \right\} \]

Using the remaining delta functions (we suppress the \(x\)-dependence, which is now trivial):

\[ [K_i, P_j] = -i \left\{ -\int d^3 x \nabla^k \phi \nabla^j \phi + \int d^3 x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3 x x^i \nabla^j \Pi \right\} \]

Now we evaluate the derivative in the first term:

\[ [K_i, P_j] = -i \left\{ -\int d^3 x x^i \nabla^j \phi \nabla^k \phi - \int d^3 x \delta^i_k \nabla^k \phi \nabla^j \phi + \int d^3 x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3 x x^i \nabla^j \Pi \right\} \]
We use the delta in the second term, and integrate by parts in the first term:

\[ [K_i, P_j] = -i \left\{ \int d^3x \nabla^k (x^i \nabla^j \phi) \nabla^k \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x \frac{\delta V}{\delta \phi} \nabla^i \phi + \int d^3x \Pi \nabla^j \phi \right\} \]

Evaluating the derivative in the first term:

\[ [K_i, P_j] = -i \left\{ \int d^3x \nabla^j \phi \nabla^k \phi + \int d^3x \nabla^i \phi \nabla^j \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x \frac{\delta V}{\delta \phi} \nabla^j \phi \right. \]

\[ \left. + \int d^3x \Pi \nabla^j \phi \right\} \]

Reversing the order of the derivatives in the first term, and using the delta function in the second term:

\[ [K_i, P_j] = -i \left\{ \int d^3x \nabla^j \phi \nabla^k \phi + \int d^3x \nabla^i \phi \nabla^j \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x \frac{\delta V}{\delta \phi} \nabla^j \phi \right. \]

\[ \left. + \int d^3x \Pi \nabla^j \phi \right\} \]

Now the second and third terms cancel:

\[ [K_i, P_j] = -i \left\{ \int d^3x \nabla^j \phi \nabla^k \phi + \int d^3x \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3x \Pi \nabla^j \phi \right\} \]

Let’s rewrite the remaining terms:

\[ [K_i, P_j] = -i \left\{ \frac{1}{2} \int d^3x \nabla^j (\Pi)^2 + \frac{1}{2} \int d^3x \nabla^i (\nabla^k \phi)^2 + \int d^3x \nabla^i \nabla^j V(\phi) \right\} \]

Now, if \( i \neq j \), then the \( x^i \) can slip inside the derivative. This would give us the integral of a total derivative, which vanishes. So, we can get a nonzero result if and only if \( i = j \). Thus:

\[ [K_i, P_j] = -i \delta_{ij} \left\{ \frac{1}{2} \int d^3x \nabla^j (\Pi)^2 + \frac{1}{2} \int d^3x \nabla^i (\nabla^k \phi)^2 + \int d^3x \nabla^i \nabla^j V(\phi) \right\} \]

Note that we must write the \( \delta_{ij} \) rather than simply setting \( i = j \), since \( i \) and \( j \) are “independent” variables, ie they are set on the left hand side of the equation. Now we can integrate by parts in all three terms, the result is:

\[ [K_i, P_j] = i \delta_{ij} \left\{ \frac{1}{2} \int d^3x (\Pi)^2 + \frac{1}{2} \int d^3x (\nabla^k \phi)^2 + \int d^3x V(\phi) \right\} \]

which is:

\[ [K_i, P_j] = i \delta_{ij} \int d^3x \left\{ \frac{1}{2} (\Pi)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right\} \]

Hence,

\[ [K_i, P_j] = i \delta_{ij} H \]

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which completes our verification of equation 2.19.

Next:

\[ [J_i, J_j] = \frac{1}{4} \varepsilon_{iab} \varepsilon_{jcd} [M^{ab}, M^{cd}] \]

\[ \Rightarrow [J_i, J_j] = \frac{1}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x d^3 y \left[ M^{0ab}(x), M^{0cd}(y) \right] \]

\[ \Rightarrow [J_i, J_j] = \frac{1}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x d^3 y \left[ x^a T^{0b}(x) - x^b T^{0a}(x), y^c T^{0d}(y) - y^d T^{0c}(y) \right] \]

\[ \Rightarrow [J_i, J_j] = \frac{1}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x d^3 y \left\{ [x^a T^{0b}(x), y^c T^{0d}(y)] + (c \leftrightarrow d) + (a \leftrightarrow b) + (a, c \leftrightarrow b, d) \right\} \]

Note that the signs of the permutations are all positive due to the Levi-Cevita symbols. From now on we’ll just write these last three terms as “perms” to save space. Then:

\[ [J_i, J_j] = \frac{1}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x d^3 y \left\{ x^a y^c \left[ T^{0b}(x), T^{0d}(y) \right] + \text{(perms)} \right\} \]

Now we use our result from part (a):

\[ [J_i, J_j] = \frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x d^3 y \left\{ x^a y^c \left[ \Pi(x) \nabla_y^b \delta^a(x - y) \nabla^d \phi(y) - \Pi(y) \nabla_x^a \delta^a(x - y) \nabla^b \phi(x) \right] + \text{(perms)} \right\} \]

Next we integrate by parts (to isolate the delta function), then use the delta function:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi - x^a x^c \nabla^d \Pi \nabla^b \phi + \text{(perms)} \right\} \]

The Levi-Cevita symbols require that \( a \neq b \) and \( c \neq d \), so:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi + x^a x^c \nabla^d \Pi \nabla^b \phi + \text{(perms)} \right\} \]

Now we integrate by parts in the second term:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi + x^a x^c \nabla^d \Pi \nabla^b \phi + \text{(perms)} \right\} \]

Since \( c \neq d \), we remove that term from the derivative:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi + \Pi x^c \nabla^d (x^a \nabla^b \phi) + \text{(perms)} \right\} \]

Using the product rule:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi + \Pi x^c x^a \nabla^d \nabla^b \phi + \Pi x^c \delta^{ad} \nabla^b \phi + \text{(perms)} \right\} \]

Now we reverse the order of the derivatives in the second term:

\[ [J_i, J_j] = -\frac{i}{4} \varepsilon_{iab} \varepsilon_{jcd} \int d^3 x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi + \Pi x^c x^a \nabla^b \nabla^d \phi + \Pi x^c \delta^{ad} \nabla^b \phi + \text{(perms)} \right\} \]
and integrate by parts in the second term, remembering that \( a \neq b \):

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{jcd} \int d^3x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi - \nabla^b (\Pi x^c) x^a \nabla^d \phi + \Pi x^c \epsilon^{ad} \nabla^b \phi \right\} (\text{perms})
\]

Using the product rule:

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{jcd} \int d^3x \left\{ x^a x^c \nabla^b \Pi \nabla^d \phi - \Pi \delta^{bc} x^a \nabla^d \phi - x^a x^c \nabla^b \Pi \nabla^d \phi + \Pi x^c \epsilon^{ad} \nabla^b \phi \right\} (\text{perms})
\]

The first and third terms vanish:

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{jcd} \int d^3x \left\{ \Pi x^c \epsilon^{ad} \nabla^b \phi - \Pi \delta^{bc} x^a \nabla^d \phi \right\} (\text{perms})
\]

Writing these terms separately:

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{jcd} \int d^3x \left\{ \Pi x^c \epsilon^{ad} \nabla^b \phi \right\} + \frac{i}{4} \epsilon_{iab} \epsilon_{jcd} \int d^3x \Pi \delta^{bc} x^a \nabla^d \phi (\text{perms})
\]

Now we use the delta functions:

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{jca} \int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4} \epsilon_{iab} \epsilon_{jbd} \int d^3x \Pi x^a \nabla^d \phi (\text{perms})
\]

Rewriting the Levi-Cevita symbols:

\[
[J_i, J_j] = -\frac{i}{4} \epsilon_{iab} \epsilon_{acj} \int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4} \epsilon_{iab} \epsilon_{bjd} \int d^3x \Pi x^a \nabla^d \phi (\text{perms})
\]

Now we use the identity for Levi-Cevita symbols:

\[
[J_i, J_j] = -\frac{i}{4} (\delta_{ic} \delta_{bj} - \delta_{ij} \delta_{bc}) \int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4} (\delta_{aj} \delta_{id} - \delta_{ad} \delta_{ij}) \int d^3x \Pi x^a \nabla^d \phi (\text{perms})
\]

Expanding this:

\[
[J_i, J_j] = \frac{i}{4} \left\{ -\int d^3x \Pi x^i \nabla^j \phi + \int d^3x \Pi x^b \nabla^b \phi + \int d^3x \Pi x^i \nabla^i \phi - \int d^3x \Pi x^a \nabla^a \phi \right\} (\text{perms})
\]

In the second and fourth terms, \( a \) and \( b \) are reduced to dummy variables, and these terms cancel. Then:

\[
[J_i, J_j] = \frac{i}{4} \left\{ -\int d^3x \Pi x^i \nabla^j \phi + \int d^3x \Pi x^i \nabla^i \phi + (\text{perms}) \right\}
\]

Remember that our permutations involve permutating \( a, b, c, \) and \( d \). But none of these terms matter anymore! So, the three permutations are identical to the first term, and:

\[
[J_i, J_j] = i \left\{ \int d^3x \left( \Pi x^i \nabla^i \phi - \Pi x^i \nabla^i \phi \right) \right\}
\]
This, of course, is $\mathcal{M}$:

$$[J_i, J_j] = -i \left\{ \int d^3 x \mathcal{M}_{0ji} \right\}$$

which is:

$$[J_i, J_j] = -i M^{ji}$$

Since $M$ is anti-symmetric:

$$[J_i, J_j] = i M^{ij}$$

Now let’s write this differently:

$$[J_i, J_j] = \frac{i}{2} (M^{ij} - M^{ji})$$

$$\implies [J_i, J_j] = \frac{i}{2} (\delta_{jb}\delta_{ia} - \delta_{ja}\delta_{ib}) M^{ab}$$

$$\implies [J_i, J_j] = \frac{i}{2} \varepsilon_{kij} \varepsilon_{kab} M^{ab}$$

$$\implies [J_i, J_j] = \frac{i}{2} \varepsilon_{ijk} \varepsilon_{kab} M^{ab}$$

which implies:

$$[J_i, J_j] = i \varepsilon_{ijk} J_k$$

Next:

$$[J_i, K_j] = \left[ \frac{1}{2} \varepsilon_{iab} M^{ab}, M^{j0} \right]$$

$$\implies [J_i, K_j] = \frac{1}{2} \varepsilon_{iab} [M^{ab}, M^{j0}]$$

$$\implies [J_i, K_j] = \frac{1}{2} \varepsilon_{iab} \int d^3 x d^3 y \left\{ \mathcal{M}^{ab}(x), \mathcal{M}^{0j}(y) \right\}$$

$$\implies [J_i, K_j] = \frac{1}{2} \varepsilon_{iab} \int d^3 x d^3 y \left\{ x^a T^{0b}(x) - x^b T^{0a}(x), x^j T^{00}(y) - x^0 T^{0j}(y) \right\}$$

Note that $x^0 = t$. Let’s choose to evaluate the commutator at $t = 0$. Since commutators should not be time-dependent, the value at $t = 0$ should equal the commutator at all other times. Hence,

$$[J_i, K_j] = \frac{1}{2} \varepsilon_{iab} \int d^3 x d^3 y \left\{ x^a x^j \left[ T^{0b}(x), T^{00}(y) \right] + (a \leftrightarrow b) \right\}$$

Using our result from part (a):

$$[J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3 x d^3 y \left\{ x^a x^j \nabla_k^k \phi(x) \nabla_k \phi(x) \nabla_k^k \delta^3(x - y) \nabla_y^b \phi(y) + x^a x^j \frac{\delta V}{\delta \phi} \delta^3(x - y) \nabla_y^b \phi(y) \right.$$

$$- x^a x^j \Pi(y) \Pi(x) \nabla_y^b \delta^3(x - y) + (a \leftrightarrow b) \right\}$$
Integrating by parts:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x d^3y \left\{ -\nabla^b_y \phi(y) \delta^3(x - y) \nabla^k_x \left[ x^a x^j \nabla^k_x \phi(x) \right] + x^a x^j \frac{\delta V}{\delta \phi} \delta^3(x - y) \nabla^b_y \phi(y) \\
+ x^a x^j \Pi(x) \nabla^k_y \Pi(y) \delta^3(x - y) + (a \leftrightarrow b) \right\} \]

Starting now we’ll suppress the permutation until the end of the problem. Using the delta functions:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -\nabla^b \phi \nabla^k \left[ x^a x^j \nabla^k \phi \right] + x^a x^j \frac{\delta V}{\delta \phi} \nabla^b \phi + x^a x^j \Pi \nabla^b \Pi \right\} \]

Now in this first term, I want to integrate by parts, reverse the order of the derivatives, then integrate by parts again. This gives:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -\nabla^k \phi \nabla^b \left[ x^a x^j \nabla^k \phi \right] + x^a x^j \frac{\delta V}{\delta \phi} \nabla^b \phi + x^a x^j \Pi \nabla^b \Pi \right\} \]

If \( b \neq k \), then this first term is the integral of a total derivative, which vanishes. So, this is nonzero only if \( b = k \), so:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -\nabla^k \phi \nabla^b \left[ x^a x^j \nabla^k \phi \right] + x^a x^j \frac{\delta V}{\delta \phi} \nabla^b \phi + x^a x^j \Pi \nabla^b \Pi \right\} \]

The Levi-Cevita symbol requires that \( a \neq b \), so:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a \nabla^b \phi \nabla^b \left[ x^j \nabla^b \phi \right] + x^a x^j \frac{\delta V}{\delta \phi} \nabla^b \phi + x^a x^j \Pi \nabla^b \Pi \right\} \]

Using the product rule:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a \nabla^b \phi \delta^{bj} \nabla^b \phi - x^a x^j \nabla^b \phi \nabla^b \nabla^b \phi + x^a x^j \frac{\delta V}{\delta \phi} \nabla^b \phi + x^a x^j \Pi \nabla^b \Pi \right\} \]

The second, third, and fourth terms can be rewritten:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a \nabla^b \phi \delta^{bj} \nabla^b \phi - \frac{1}{2} x^a x^j \nabla^b (\nabla^b \phi)^2 + x^a x^j \nabla^b V(\phi) + \frac{1}{2} x^a x^j \nabla^b (\Pi)^2 \right\} \]

Now \( a \neq b \), so:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a \nabla^b \phi \delta^{bj} \nabla^b \phi - \frac{1}{2} x^a x^j \nabla^b [x^a (\nabla^b \phi)^2] + x^a x^j \nabla^b V(\phi) + \frac{1}{2} x^a x^j \nabla^b (\Pi)^2 \right\} \]

The second term vanishes if \( b \neq j \). So:

\[ [J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a \nabla^b \phi \delta^{bj} \nabla^b \phi - \frac{1}{2} \delta^{bj} x^a x^b [x^a (\nabla^b \phi)^2] + x^a x^j \nabla^b V(\phi) + \frac{1}{2} x^a x^j \nabla^b (\Pi)^2 \right\} \]
Now let’s integrate by parts in the second, third, and fourth terms, recalling that $a \neq b$:

$$[J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -x^a (\nabla \phi)^2 \delta^{bij} + \frac{1}{2} \delta^{bij} x^a (\nabla \phi)^2 - x^a \delta^{bij} V(\phi) - \frac{1}{2} x^a \delta^{bij} (\Pi)^2 \right\}$$

Now these first two terms combine:

$$[J_i, K_j] = \frac{i}{2} \varepsilon_{iab} \int d^3x \left\{ -\frac{1}{2} x^a (\nabla \phi)^2 \delta^{bij} - x^a \delta^{bij} V(\phi) - \frac{1}{2} x^a \delta^{bij} (\Pi)^2 \right\}$$

Using the deltas:

$$[J_i, K_j] = \frac{i}{2} \varepsilon_{ija} \int d^3x \left\{ -\frac{1}{2} x^a (\Pi)^2 - \frac{1}{2} x^a (\nabla \phi)^2 - x^a V(\phi) \right\}$$

Reversing the Levi-Cevita symbol, and re-inserting the permutations:

$$[J_i, K_j] = \frac{i}{2} \varepsilon_{ija} \int d^3x \left\{ \frac{1}{2} x^a (\Pi)^2 + \frac{1}{2} x^a (\nabla \phi)^2 + x^a V(\phi) + (a \leftrightarrow b) \right\}$$

Now b is gone and a is just a dummy variable, so the permutations is equal to the original term:

$$[J_i, K_j] = i \varepsilon_{ija} \int d^3x \left\{ \frac{1}{2} x^a (\Pi)^2 + \frac{1}{2} x^a (\nabla \phi)^2 + x^a V(\phi) \right\}$$

Let’s also rename the dummy variable, $a \rightarrow k$:

$$[J_i, K_j] = i \varepsilon_{ijk} \int d^3x \left\{ \frac{1}{2} x^k (\Pi)^2 + \frac{1}{2} x^k (\nabla \phi)^2 + x^k V(\phi) \right\}$$

This obviously looks like the Hamiltonian density, so:

$$[J_i, K_j] = i \varepsilon_{ijk} \int d^3x [x^k T^{00}(x)]$$

Remembering that this should be time-independent, we know that any time-dependent term should vanish. So:

$$[J_i, K_j] = i \varepsilon_{ijk} \int d^3x [x^k T^{00}(x) - x^0 T^{0k}(x)]$$

which is equal to:

$$[J_i, K_j] = i \varepsilon_{ijk} \int d^3x M^{0k}(x)$$

which is:

$$[J_i, K_j] = i \varepsilon_{ijk} M^{k0}$$

and so:

$$[J_i, K_j] = i \varepsilon_{ijk} K_k$$

Finally:

$$[K_i, K_j] = [M^{i0}, M^{j0}]$$
\[ [K_i, K_j] = \int d^3x d^3y \left[ M^{0i0}, M^{0j0} \right] \]

\[ [K_i, K_j] = \int d^3x d^3y \left[ x^i T^{00}(x), y^j T^{00}(y) \right] \]

where, as before, we neglected those terms with explicit time-dependence, choosing to work where \( t = 0 \).

\[ [K_i, K_j] = \int d^3x d^3y x^i y^j \left[ T^{00}(x), T^{00}(y) \right] \]

Now this looks very familiar to our expression for \( [K_i, H] \). In fact, we can insert the \( x^j \) into equation (22.3.1), renaming the dummy index:

\[ [K_i, K_j] = i \int d^3x \left\{ \nabla^k (x^j \nabla^k \phi) \Pi x^i - \nabla^k (x^i \nabla^k \phi) \Pi x^j \right\} \]

Using the product rule:

\[ [K_i, K_j] = i \int d^3x \left\{ \nabla^k \phi \delta^{jk} \Pi x^i + x^j \nabla^2 \phi \Pi x^i - \nabla^k \phi \delta^{ik} \Pi x^j - x^i \nabla^2 \phi \Pi x^j \right\} \]

The second and fourth terms vanish:

\[ [K_i, K_j] = i \int d^3x \left\{ \nabla^k \phi \delta^{jk} \Pi x^i - \nabla^k \phi \delta^{ik} \Pi x^j \right\} \]

Using the deltas:

\[ \Rightarrow [K_i, K_j] = i \int d^3x \left\{ \nabla^j \phi \Pi x^i - \nabla^i \phi \Pi x^j \right\} \]

\[ \Rightarrow [K_i, K_j] = i \int d^3x \left\{ -T^{0j} x^i + T^{0i} x^j \right\} \]

\[ \Rightarrow [K_i, K_j] = -i \int d^3x \left\{ T^{0j} x^i - T^{0i} x^j \right\} \]

\[ \Rightarrow [K_i, K_j] = -i \int d^3x M^{0ij} \]

\[ \Rightarrow [K_i, K_j] = -i M^{ij} \]

Now let’s write this differently

\[ [K_i, K_j] = -i M^{ij} - M^{ji} \]

\[ \Rightarrow [K_i, K_j] = -\frac{i}{2} (\delta_{jb} \delta_{ia} - \delta_{ja} \delta_{ib}) M^{ab} \]

\[ \Rightarrow [K_i, K_j] = -\frac{i}{2} \varepsilon_{kij} \varepsilon_{kab} M^{ab} \]

\[ \Rightarrow [K_i, K_j] = -\frac{i}{2} \varepsilon_{ijk} \varepsilon_{kab} M^{ab} \]

which implies:

\[ [K_i, K_j] = -i \varepsilon_{ijk} J_k \]

which completes our verification of equation 2.17.