Srednicki Chapter 22

QFT Problems & Solutions

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Srednicki 22.1. For the Noether current of equation 22.6 and assuming that $\delta\phi$ does not involve time derivatives, use the canonical commutation relations to show that $[\phi, Q] = i\delta\phi$.

Using equation 22.17:

$$[\phi, Q] = [\phi(x), \int d^3x j^0(x')]$$

Using equation 22.6:

$$[\phi, Q] = \int d^3x [\phi(x), \frac{\partial \mathcal{L}(x')}{\partial(\partial_0 \phi_b(x'))} \delta\phi_b(x')]$$

Using the definition of the conjugate momentum:

$$[\phi, Q] = \int d^3x [\phi(x), \Pi_b(x')\delta\phi_b(x')]$$

Since $\delta\phi$ doesn't have any time derivatives, we expect that it will commute with ϕ. Then:

$$[\phi, Q] = \int d^3x [\phi(x), \Pi_b(x')]\delta\phi_b(x')$$

Now we use equation 3.28.

$$[\phi, Q] = \int d^3x i\delta^3(x - x')\delta_{ab}\delta\phi_b(x')$$

which gives:

$$[\phi, Q] = i\delta\phi$$

as expected. Two notes, both about equation 3.28:

- We assume that these are both taken at equal times, ie that $x^0 = x'^0$. This is fine because Q should be constant in time, per the discussion in the text.
- We assume that the indices a and b are equal. If not, then we have two different fields, which of course commute.

Srednicki 22.2. Use the canonical commutation relations to verify equation 22.38.

Using equation 22.35:

$$[\phi(x), P^\mu] = \int d^3x'[\phi(x), T^{0\mu}(x')] \tag{22.2.1}$$

Using equation 22.33:

$$[\phi(x), P^j] = -\int d^3x'[\phi(x), \Pi(x')\nabla^j\phi(x')]$$

We can take the $\nabla^j\phi(x')$ out of the commutator, since it obviously commutes with $\phi(x)$. Then,

$$[\phi(x), P^j] = -\int d^3x'[\phi(x), \Pi(x')]\nabla^j\phi(x')$$

Now we use the commutation relations.

$$[\phi(x), P^j] = -i\int d^3x'\delta^3(x - x')\nabla^j\phi(x')$$

And we do the integral:

$$[\phi(x), P^j] = \frac{1}{i}\nabla^j\phi(x) \tag{22.2.2}$$

Now we go back to equation (22.2.1), this time choosing to insert 22.32:

$$[\phi(x), P^0] = \int d^3x'[\phi(x), \frac{1}{2}\Pi^2(x') + \frac{1}{2}(\nabla\phi)^2 + V(\phi)]$$

Note that we have implicitly set the right side to have a subscripts, since any other subscript refers to a different particle, and therefore commutes. Now all these terms on the right hand side commute with ϕ except the second one. So,

$$[\phi(x), P^0] = \frac{1}{2}\int d^3x'[\phi(x), \Pi^2(x')]$$

which gives, using the properties of commutators:

$$[\phi(x), P^0] = \frac{1}{2}\int d^3x'[\phi(x), \Pi(x')]\Pi(x') + \frac{1}{2}\int d^3x'\Pi(x')[\phi(x), \Pi(x')]$$

From which it follows:

$$[\phi(x), P^0] = \int d^3x' i\delta^3(x - x')\Pi(x')$$

Doing the integral:

$$[\phi(x), P^0] = -\frac{1}{i}\Pi(x)$$

Now remember that Π is the time derivative of ϕ, so:

$$[\phi(x), P^0] = -\frac{1}{i}\frac{d}{dt}\phi(x) \tag{22.2.3}$$

Combining (22.2.3) and (22.2.2) into one equation, we have:

$$[\phi(x), P^0] = \frac{1}{i}\partial^\mu \phi(x)$$

as expected.

Srednicki 22.3. (a) With $T^{\mu\nu}$ given by equation 22.31, compute the equal-time commutators $[T^{00}(x), T^{00}(y)]$, $[T^{0i}(x), T^{00}(y)]$, $[T^{0i}(x), T^{0j}(y)]$.

This is a very long problem, so I want to note from the beginning that I will set $\phi_a = \phi$, since all commutators vanish if they compare two different scalar fields. I also write the dependent variables only when there is a potential ambiguity. Equation 22.31 gives:

$$[T^{00}(x), T^{00}(y)] = \left[\frac{1}{2}\Pi^2(x) + \frac{1}{2}(\nabla\phi(x))^2 + V(\phi(x)), \frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla\phi(y))^2 + V(\phi(y))\right]$$

Note that we have $(\nabla\phi)^2 = \nabla^j\phi\nabla_j\phi$. There are two different $(\nabla\phi)^2$ terms (one on each side), so we should write both of them out, using different indices to represent summation. But note that the terms where the two ∇ terms meet will all vanish (by equation 2.38), so it is simplest to just set $\nabla^i = \nabla^j = \nabla$; all the ∇s will have the same index in the end.

$$[T^{00}(x), T^{00}(y)] = \frac{1}{4}[\Pi^2(x), (\nabla\phi(y))^2] + \frac{1}{2}[\Pi^2(x), V(\phi(y))] - \frac{1}{4}[\Pi^2(y), (\nabla\phi(x))^2] - \frac{1}{2}[\Pi^2(y), V(\phi(x))]$$

where we have neglected the terms which vanish (recalling that the canonical commutation relations require that Π and ϕ commute with themselves, even if the variables are different). Next we use the properties of the commutators:

$$[T^{00}(x), T^{00}(y)] = \frac{1}{4}[\Pi^2(x), \nabla\phi(y)]\nabla\phi(y) + \frac{1}{4}\nabla\phi(y)[\Pi^2(x), \nabla\phi(y)] + \frac{1}{2}[\Pi^2(x), V(\phi(y))]$$

$$-\frac{1}{4}[\Pi^2(y), \nabla\phi(x)]\nabla\phi(x) - \frac{1}{4}\nabla\phi(x)[\Pi^2(y), \nabla\phi(x)] - \frac{1}{2}[\Pi^2(y), V(\phi(x))]$$

Most of these terms have a nabla (spatial derivative). This can be moved outside the integral, since the spatial variable in quantum field theory is just a label, not an operator. Then:

$$[T^{00}(x), T^{00}(y)] = \frac{1}{4}\nabla_y[\Pi^2(x), \phi(y)]\nabla\phi(y) + \frac{1}{4}\nabla_y\phi(y)\nabla[\Pi^2(x), \phi(y)] + \frac{1}{2}[\Pi^2(x), V(\phi(y))]$$

$$-\frac{1}{4}\nabla_x[\Pi^2(y), \phi(x)]\nabla\phi(x) - \frac{1}{4}\nabla\phi(x)\nabla_x[\Pi^2(y), \phi(x)] - \frac{1}{2}[\Pi^2(y), V(\phi(x))]$$

We can easily show that $[\Pi^2(x), \phi(y)] = -2i\delta^3(x-y)\Pi(x)$ [we also showed this in problem 22.2]. Then:

$$[T^{00}(x), T^{00}(y)] = \frac{1}{2i}\nabla_y\delta^3(x-y)\Pi(x)\nabla\phi(y) + \frac{1}{2i}\nabla\phi(y)\nabla_y\delta^3(x-y)\Pi(x) + \frac{1}{2}[\Pi^2(x), V(\phi(y))]$$

$$-\frac{1}{2i}\nabla_x\delta^3(x-y)\Pi(y)\nabla\phi(x) - \frac{1}{2i}\nabla\phi(x)\nabla_x\delta^3(x-y)\Pi(y) - \frac{1}{2}[\Pi^2(y), V(\phi(x))]$$

Now, what to do with $V(\phi(x))$? Whatever it is, let's do a Taylor expansion, so we have

$$[\Pi^2(y), V(\phi(x))] = \sum_n [\Pi^2(y), c_n \phi(x)^n]$$

Now our properties of commutators give us

$$\sum_n n c_n \phi(x)^{i-1} [\Pi^2(y), \phi(x)]$$

from which we have, remembering that ϕ always commutes with itself,

$$\frac{1}{i} \sum_n n c_n \phi(x)^{i-1} \delta^3(x-y) \Pi(x) + \frac{1}{i} \sum_n n c_n \Pi(x) \phi(x)^{i-1} \delta^3(x-y)$$

Notice that we can write this as

$$\frac{1}{i} \Pi(x) \frac{\delta V(\phi(x))}{\delta \phi} \delta^3(x-y) + \frac{1}{i} \frac{\delta V(\phi(x))}{\delta \phi} \delta^3(x-y) \Pi(x)$$

By the way, this highly useful derivation is absent from most lists of identities, presumably because our assumption that all functions can be written as a Taylor Series is questionable (but it's OK for our purposes, our potentials should be polynomial functions already, certainly nothing more complicated than an exponential). Using this in our expression for $[T^{00}(x), T^{00}(y)]$, we have:

$$[T^{00}(x), T^{00}(y)] = \frac{1}{2i} \nabla_y \delta^3(x-y) \Pi(x) \nabla \phi(y) + \frac{1}{2i} \nabla \phi(y) \nabla_y \delta^3(x-y) \Pi(x) + \frac{1}{2i} \frac{\delta V}{\delta \phi} \delta^3(x-y) \Pi(x)$$

$$+ \frac{1}{2i} \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x-y) - \frac{1}{2i} \nabla_x \delta^3(x-y) \Pi(y) \nabla \phi(x) - \frac{1}{2i} \nabla \phi(x) \nabla_x \delta^3(x-y) \Pi(y) - \frac{1}{2i} \frac{\delta V}{\delta \phi} \delta^3(x-y) \Pi(y)$$

$$- \frac{1}{2i} \Pi(y) \frac{\delta V}{\delta \phi} \delta^3(x-y)$$

Finally, we rewrite this a bit, and reinsert our nabla product index for clarity:

$$\boxed{\begin{aligned} [T^{00}(x), T^{00}(y)] = -\tfrac{i}{2} \Pi(x) \left(\nabla^j \phi(y) \nabla^j_y + \tfrac{\delta V}{\delta \phi} \right) \delta^3(x-y) + \tfrac{i}{2} \Pi(y) \left(\nabla^j \phi(x) \nabla^j_x + \tfrac{\delta V}{\delta \phi} \right) \delta^3(x-y) \\ - \tfrac{i}{2} \left(\nabla^j \phi(y) \nabla^j_y + \tfrac{\delta V}{\delta \phi} \right) \delta^3(x-y) \Pi(x) + \tfrac{i}{2} \left(\nabla^j \phi(x) \nabla^j_x + \tfrac{\delta V}{\delta \phi} \right) \delta^3(x-y) \Pi(y) \end{aligned}}$$

Next:

$$[T^{00}(x), T^{0i}(y)] = \left[\frac{1}{2} \Pi^2(x) + \frac{1}{2} \nabla \phi(x) + V(\phi), -\Pi(y) \nabla \phi \right]$$

As before, we use the commutation rules to separate this, using j superscripts to indicate multiplication where needed. We achieve:

$$[T^{00}(x), T^{0i}(y)] = \frac{1}{2} \Pi(y) \Pi^j(x) \nabla^i_y \left[\phi(y), \Pi^j(x) \right] + \frac{1}{2} \Pi(y) \nabla^i_y \left[\phi(y), \Pi^j(x) \right] \Pi^j(x)$$

$$- \frac{1}{2} \nabla^j_x \left[\phi(x), \Pi, y) \right] \nabla^j \phi(x) \nabla^i \phi(y) - \frac{1}{2} \nabla^j \phi(x) \nabla^j_x \left[\phi(x), \Pi(y) \right] \nabla^i \phi(y)$$

$$+\Pi(y)\nabla^i\left[\phi(y), V(\phi)\right] + \left[\Pi(y), V(\phi)\right]\nabla^i\phi(y)$$

The first term on the third line vanishes because ϕ always commutes with itself. The last term is deconstructed into a Taylor series as before. The first four terms can be calculated directly. The result is:

$$[T^{00}(x), T^{0i}(y)] = \frac{i}{2}\Pi(y)\Pi(x)\nabla_y^i\delta^3(x-y) + \frac{i}{2}\Pi(y)\left\{\nabla_y^i\delta^3(x-y)\right\}\Pi(x)$$

$$-\frac{i}{2}\left\{\nabla_x^j\delta^3(x-y)\right\}\nabla^j\phi(x)\nabla^i\phi(y) - \frac{i}{2}\nabla^j\phi(x)\left\{\nabla_x^j\delta^3(x-y)\right\}\nabla^i\phi(y) - i\frac{\delta V}{\delta\phi}\nabla^i\phi(y)\delta^3(x-y)$$

Note that a few of the j superscripts (indicating multiplication) disappeared, since multiplying any operator by the delta function is assumed. Cleaning this up, we have:

$$\boxed{[T^{00}(x), T^{0i}(y)] = -i\left[\nabla_x^j\phi(x)\nabla_x^j + \frac{\delta V}{\delta\phi}\right]\delta^3(x-y)\nabla_y^i\phi(y) + i\Pi(y)\Pi(x)\nabla_y^i\delta^3(x-y)}$$

Finally:

$$[T^{0i}, T^{0j}(y)] = \left[\Pi(x)\nabla^i\phi(x), \Pi(y)\nabla^j\phi(y)\right]$$

Using the commutator rules:

$$[T^{0i}, T^{0j}(y)] = \Pi(x)\left\{\nabla_x^i\left[\phi(x), \Pi(y)\right]\right\}\nabla^j\phi(y) + \Pi(y)\left\{\nabla_y^j\left[\Pi(x), \phi(y)\right]\right\}\nabla^i\phi(x)$$

Now we use the commutation relations (equation 3.28):

$$\boxed{[T^{0i}, T^{0j}(y)] = i\Pi(x)\nabla_x^i\delta^3(x-y)\nabla^j\phi(y) - i\Pi(y)\nabla_y^j\delta^3(x-y)\nabla^i\phi(x)}$$

(b) Use your results to verify equations 2.17, 2.19, and 2.20.

We now have to verify nine equations. Let's start with this one:

$$[P_i, H] = \left[\int d^3x T^{0i}(x), \int d^3y \Pi^2(y) + \frac{1}{2}(\nabla\phi(y))^2 + V(\phi(y))\right]$$

which gives:

$$[P_i, H] = \int d^3x d^3y \left[T^{0i}(x), T^{00}(y)\right]$$

We worked out this commutator in part (a), the result is:

$$[P_i, H] = \int d^3x d^3y \left\{i\left[\nabla_x^j\phi(x)\nabla_x^j + \frac{\delta V}{\delta\phi}\right]\delta^3(x-y)\nabla_y^i\phi(y) - i\Pi(y)\Pi(x)\nabla_y^i\delta^3(x-y)\right\}$$

Breaking this up and doing some integration by parts (remember that integration by parts introduces a negative sign and shifts, for example, ∇_x from its current location to all other functions of x):

$$[P_i, H] = i\int d^3x d^3y \left[-\nabla_x^j\nabla_x^j\phi(x)\delta^3(x-y)\nabla_y^i\phi(y)\right] + i\frac{\delta V}{\delta\phi}\nabla_y^i\phi(y) + i\int d^3x d^3y \left(\nabla_y^i\Pi(y)\right)\Pi(x)\delta^3(x-y)$$

Using the delta functions, we have:

$$[P_i, H] = -i \int d^3y \left[\nabla^2\phi(y)\nabla^i\phi(y) - i\nabla^i\Pi(y)\Pi(y) - \frac{\delta V}{\delta\phi}\nabla^i\phi(y)\right]$$

Now, note that we can rewrite this:

$$[P_i, H] = i \int d^3y \nabla^i \left[\frac{1}{2}\nabla^j\phi\nabla^j\phi + \frac{1}{2}\Pi(y)\Pi(y) + V(\phi)\right]$$

[If this first term does not seem obvious, use the product rule, then reverse the order of the derivatives (they commute) and integrate by parts.] Now remember that the integral of a total derivative vanishes assuming suitable boundary conditions at infinity (by the chain rule). So,

$$\boxed{[P_i, H] = 0}$$

as expected.

Second:

$$[P_i, P_j] = \int d^3x d^3y \left[T^{0i}(x), T^{0j}(y)\right]$$

Using our result from part (a), we have:

$$[P_i, P_j] = \int d^3x d^3y \left[i\Pi(x)\nabla^i_x\delta^3(x-y)\nabla^j\phi(y) - i\Pi(y)\nabla^j_y\delta^3(x-y)\nabla^i\phi(x)\right]$$

Integrating by parts:

$$[P_i, P_j] = i \int d^3x d^3y \left[-\nabla^i_x\Pi(x)\nabla^j\phi(y)\delta^3(x-y) + \nabla^j_y\Pi(y)\nabla^i\phi(x)\delta^3(x-y)\right]$$

Using the delta functions:

$$[P_i, P_j] = i \int d^3x \left[-\nabla^i\Pi\nabla^j\phi + \nabla^j\Pi\nabla^i\phi\right]$$

Now we can integrate by parts on the last term, commute the two derivatives, then integrate by parts again. This causes the two terms to cancel, giving:

$$\boxed{[P_i, P_j] = 0}$$

This completes our verification of equation 2.20.

Next, we'll calculate:

$$[J_i, H] = \int d^3y \left[\frac{1}{2}\varepsilon_{ijk}M^{jk}, T^{00}(y)\right]$$

$$\implies [J_i, H] = \frac{1}{2}\varepsilon_{ijk} \int d^3x d^3y \left[\mathcal{M}^{0jk}(x), T^{00}(y)\right]$$

$$\implies [J_i, H] = \frac{1}{2}\varepsilon_{ijk}\int d^3x d^3y \left[x^j T^{0k}(x) - x^k T^{0j}(x), T^{00}(y)\right]$$

$$\implies [J_i, H] = \frac{1}{2}\varepsilon_{ijk}\int d^3x d^3y x^j \left[T^{0k}(x), T^{00}(y)\right] + \frac{1}{2}\varepsilon_{ikj}\int d^3x d^3y x^k \left[T^{0j}(x), T^{00}(y)\right]$$

Does this integral seem familiar? It's exactly what we got when we calculated $[P_i, H]$, only there's an additional term of x^j (shift the indices needed for the different terms in the expression). When we do the integration by parts, we will have something equivalent to $\nabla_x^j \left(x^k \nabla^j \phi(x)\right)$. But, notice that our levi-cevita symbol (ε_{ijk}) requires that j and k be different variables. The result is that the x^k does not affect anything (it is a constant with respect to the derivative), and ends up as another term inside the constant derivative:

$$[J_i, H] = \frac{i}{2}\varepsilon_{ijk}\int d^3y \nabla^i \left[y^j \frac{1}{2}\nabla^k\phi\nabla^k\phi + y^j\frac{1}{2}\Pi(y)\Pi(y) + y^j V(\phi)\right] + (j \leftrightarrow k)$$

which, as before, gives zero:

$$\boxed{[J_i, H] = 0}$$

Next:

$$[J_i, P_j] = \left[\frac{1}{2}\varepsilon_{i\ell k}M^{\ell k}, \int d^3y T^{0j}(y)\right]$$

which gives:

$$[J_i, P_j] = \frac{1}{2}\varepsilon_{i\ell k}\int d^3x d^3y \left[\mathcal{M}^{0\ell k}(x), \int T^{0j}(y)\right]$$

$$\implies [J_i, P_j] = \frac{1}{2}\varepsilon_{i\ell k}\int d^3x d^3y \left[x^\ell T^{0k}(x) - x^k T^{0\ell}(x), \int T^{0j}(y)\right]$$

$$\implies [J_i, P_j] = \frac{1}{2}\varepsilon_{i\ell k}\int d^3x d^3y x^e ll \left\{\left[T^{0k}(x), \int T^{0j}(y)\right] + (k \leftrightarrow \ell)\right\}$$

Note that the sign is positive due to the Levi-Cevita symbol. Using our result from part (a), we have:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x d^3y x^e ll \left\{\Pi(x)\nabla_x^k\delta^3(x-y)\nabla^j\phi(y) - \Pi(y)\nabla_y^j\delta^3(x-y)\nabla^k\phi(x) + (k \leftrightarrow \ell)\right\}$$

Integrating by parts:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x d^3y \left\{x^\ell\nabla^j\Pi(y)\nabla^k\phi(x)\delta^3(x-y) - \nabla^k\left(x^\ell\Pi(x)\right)\nabla^j\phi(y)\delta^3(x-y) + (k \leftrightarrow \ell)\right\}$$

Using the delta function:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x \left[x^\ell\nabla^j\Pi\nabla^k\phi - \nabla^k\left(x^\ell\Pi\right)\nabla^j\phi + (k \leftrightarrow \ell)\right]$$

Evaluating the derivative in the second term, we have:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x \left[x^\ell\nabla^j\Pi\nabla^k\phi - x^\ell\nabla^k\Pi\nabla^j\phi - \delta^{k\ell}\Pi\nabla^j\phi + (k \leftrightarrow \ell)\right]$$

Now we use integration by parts in the second term:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x\left[x^\ell\nabla^j\Pi\nabla^k\phi + \nabla^k(x^\ell\nabla^j\phi)\Pi - \delta^{k\ell}\Pi\nabla^j\phi + (k\leftrightarrow\ell)\right]$$

When evaluating this derivative with the product rule, one of the terms cancels with the last term. So:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x\left[x^\ell\nabla^j\Pi\nabla^k\phi + x^\ell\nabla^k(\nabla^j\phi)\Pi + (k\leftrightarrow\ell)\right]$$

Reversing the order of the two derivatives:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x\left[x^\ell\nabla^j\Pi\nabla^k\phi + x^\ell\nabla^j(\nabla^k\phi)\Pi + (k\leftrightarrow\ell)\right]$$

Integrating by parts again:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x\left[x^\ell\nabla^j\Pi\nabla^k\phi - \nabla^j(x^\ell\Pi)\nabla^k\phi + (k\leftrightarrow\ell)\right]$$

Doing the derivative this time, only one term remains:

$$[J_i, P_j] = \frac{i}{2}\varepsilon_{i\ell k}\int d^3x\left[-\Pi\delta^{j\ell}\nabla^k\phi + (k\leftrightarrow\ell)\right]$$

which gives (remembering that the plus sign indicates that the second term should have the same sign as the first term):

$$[J_i, P_j] = \frac{i}{2}\int d^3x\left[-\varepsilon_{i\ell k}\Pi\delta^{j\ell}\nabla^k\phi - \varepsilon_{ik\ell}\Pi\delta^{jk}\nabla^\ell\phi\right]$$

Using the Levi-Cevita symbol:

$$[J_i, P_j] = \frac{i}{2}\int d^3x\left[-\varepsilon_{i\ell k}\Pi\delta^{j\ell}\nabla^k\phi - \varepsilon_{ik\ell}\Pi\delta^{jk}\nabla^\ell\phi\right]$$

Using the delta function:

$$[J_i, P_j] = \frac{i}{2}\int d^3x\left[-\varepsilon_{ijk}\Pi\nabla^k\phi - \varepsilon_{ij\ell}\Pi\nabla^\ell\phi\right]$$

Notice that ℓ is now just a dummy variable, just like k. Let's redefine for consistency $\ell\to k$. Then,

$$[J_i, P_j] = -i\varepsilon_{ijk}\int d^3x\Pi\nabla^k\phi$$

which gives:

$$\boxed{[J_i, P_j] = i\varepsilon_{ijk}P^k}$$

Then:

$$[K_i, H] = [M^{i0}, H]$$

$$\implies [K_i, H] = \left[\int d^3x\mathcal{M}^{0i0}(x), \int d^3yT^{00}(y)\right]$$

$$\implies [K_i, H] = \int d^3x d^3y \left[x^i T^{00}(x) - x^0 T^{0i}(x), T^{00}(y)\right]$$

Note that $x^0 = t$. Let's choose to evaluate the commutator at $t = 0$. Since commutators should not be time-dependent, the value at $t = 0$ should equal the commutator at all other times. Hence,

$$[K_i, H] = \int d^3x d^3y x^i \left[T^{00}(x), T^{00}(y)\right]$$

Using our result from part (a):

$$[K_i, H] = \frac{i}{2} \int d^3x d^3y \Big\{ -x^i \Pi(x) \nabla^j \phi(y) \nabla^j_y \delta^3(x-y) - x^i \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x-y) + \Pi(y) x^i \nabla^j \phi(x) \nabla^j_x \delta^3(x-y)$$

$$+ \Pi(y) x^i \frac{\delta V}{\delta \phi} \delta^3(x-y) - \nabla^j \phi(y) \nabla^j_y \delta^3(x-y) \Pi(x) x^i - \frac{\delta V}{\delta \phi} \delta^3(x-y) \Pi(x) x^i + x^i \nabla^j \phi(x) \nabla^j_x \delta^3(x-y) \Pi(y)$$

$$+ \frac{\delta V}{\delta \phi} \delta^3(x-y) \Pi(y) x^i \Big\}$$

Splitting up this integral, and integrating by parts as necessary:

$$[K_i, H] = \frac{i}{2} \Big\{ \int d^3x d^3y \left(\nabla^2 \phi(y) \Pi(x) x^i \delta^3(x-y)\right) + \int d^3x d^3y \left(-x^i \Pi(x) \frac{\delta V}{\delta \phi} \delta^3(x-y)\right)$$

$$+ \int d^3x d^3y \left(-\Pi(y) \nabla^j_x \left[x^i \nabla^j \phi(x)\right] \delta^3(x-y)\right) + \int d^3x d^3y \left(\Pi(y) x^i \frac{\delta V}{\delta \phi} \delta^3(x-y)\right)$$

$$+ \int d^3x d^3y \left(\nabla^2 \phi(y) \Pi(x) x^i \delta^3(x-y)\right) + \int d^3x d^3y \left(-x^i \frac{\delta V}{\delta \phi} \Pi(x) \delta^3(x-y)\right)$$

$$+ \int d^3x d^3y \left(-\nabla^j_x \left[x^i \nabla^j \phi(x)\right] \Pi(y) \delta^3(x-y)\right) + \int d^3x d^3y \left(x^i \Pi(y) \frac{\delta V}{\delta \phi} \delta^3(x-y)\right) \Big\}$$

Using the delta functions, and suppressing the variable (x), which is now the only variable, we have:

$$[K_i, H] = \frac{i}{2} \Big\{ \int d^3x \nabla^2 \phi \Pi x - \int d^3x x^i \Pi \frac{\delta V}{\delta \phi} - \int d^3x \Pi \left(x^i \nabla^2 \phi + \nabla^j \phi \delta^{ij}\right) + \int d^3x \Pi x^i \frac{\delta V}{\delta \phi}$$

$$\int d^3x \nabla^2 \phi \Pi x^i - \int d^3x \Pi \frac{\delta V}{\delta \phi} x^i - \int d^3x \Pi \left(x^i \nabla^2 \phi + \delta^{ij} \nabla^j \phi\right) + \int d^3x x^i \Pi \frac{\delta V}{\delta \phi} \Big\}$$

The second, fourth, sixth, and eighth terms cancel. Two of the remaining terms combine. We're left with:

$$[K_i, H] = i \int d^3x \nabla^j (\nabla^j \phi) \Pi x^i - \nabla^j (x^i \nabla^j \phi) \Pi \tag{22.3.1}$$

The first and fifth terms cancel with the first term in the third and seventh terms. The two remaining terms combine to give:

$$[K_i, H] = -i \int d^3x \Pi \delta^{ij} \nabla^j \phi$$

$$\implies [K_i, H] = -i \int d^3x \Pi \nabla^i \phi$$

$$\implies [K_i, H] = i \int d^3x T^{0i}(x)$$

where we reinserted the variable (x), giving:

$$\boxed{[K_i, H] = iP^i}$$

Next:

$$[K_i, P_j] = \left[M^{i0}, \int d^3y T^{0j}(y)\right]$$

$$\implies [K_i, P_j] = \left[\int d^3x M^{0i0}(x), \int d^3y T^{0j}(y)\right]$$

$$\implies [K_i, P_j] = \int d^3x d^3y \left[x^i T^{00}(x) - x^0 T^{0i}(x), T^{0j}(y)\right]$$

Note that $x^0 = t$. Let's choose to evaluate the commutator at $t = 0$. Since commutators should not be time-dependent, the value at $t = 0$ should equal the commutator at all other times. Hence,

$$[K_i, P_j] = \int d^3x d^3y x^i \left[T^{00}(x), T^{0j}(y)\right]$$

Using our results from part a, we have:

$$[K_i, P_j] = -i \left\{ \int d^3x d^3y x^i \nabla_x^k \phi(x) \nabla_x^k \delta^3(x-y) \nabla_y^j \phi(y) + \int d^3x d^3y x^i \frac{\delta V}{\delta \phi} \delta^3(x-y) \nabla_y^j \phi(y) \right.$$

$$\left. - \int d^3x d^3y x^i \Pi(y) \Pi(x) \nabla_y^j \delta^3(x-y) \right\}$$

Now we integrate by parts in the first and third terms, and use the delta function in the second term. The result is:

$$[K_i, P_j] = -i \left\{ - \int d^3x d^3y \nabla_x^k \left(x^i \nabla_x^k \phi(x)\right) \delta^3(x-y) \nabla_y^j \phi(y) + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi \right.$$

$$\left. \int d^3x d^3y x^i \nabla_y^j \Pi(y) \Pi(x) \delta^3(x-y) \right\}$$

Using the remaining delta functions (we supress the x-dependence, which is now trivial):

$$[K_i, P_j] = -i \left\{ - \int d^3x \nabla^k \left(x^i \nabla^k \phi\right) \nabla^j \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3x x^i \Pi \nabla^j \Pi \right\}$$

Now we evaluate the derivative in the first term:

$$[K_i, P_j] = -i \left\{ - \int d^3x x^i \nabla^j \phi \nabla^k \nabla^k \phi - \int d^3x \delta^{ik} \nabla^k \phi \nabla^j \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3x x^i \Pi \nabla^j \Pi \right\}$$

We use the delta in the second term, and integrate by parts in the first term:

$$[K_i, P_j] = -i\left\{\int d^3x \nabla^k \left(x^i \nabla^j \phi\right) \nabla^k \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3x x^i \Pi \nabla^j \Pi\right\}$$

Evaluating the derivative in the first term:

$$[K_i, P_j] = -i\left\{\int d^3x x^i \nabla^k \nabla^j \phi \nabla^k \phi + \int d^3x \nabla^j \phi \delta^{ik} \nabla^k \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi \right.$$
$$\left. + \int d^3x x^i \Pi \nabla^j \Pi\right\}$$

Reversing the order of the derivatives in the first term, and using the delta function in the second term:

$$[K_i, P_j] = -i\left\{\int d^3x x^i \nabla^j \nabla^k \phi \nabla^k \phi + \int d^3x \nabla^j \phi \nabla^i \phi - \int d^3x \nabla^i \phi \nabla^j \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi \right.$$
$$\left. + \int d^3x x^i \Pi \nabla^j \Pi\right\}$$

Now the second and third terms cancel:

$$[K_i, P_j] = -i\left\{\int d^3x x^i \nabla^j \nabla^k \phi \nabla^k \phi + \int d^3x x^i \frac{\delta V}{\delta \phi} \nabla^j \phi + \int d^3x x^i \Pi \nabla^j \Pi\right\}$$

Let's rewrite the remaining terms:

$$[K_i, P_j] = -i\left\{\frac{1}{2}\int d^3x x^i \nabla^j (\Pi)^2 + \frac{1}{2}\int d^3x x^i \nabla^j (\nabla^k \phi)^2 + \int d^3x x^i \nabla^j V(\phi)\right\}$$

Now, if $i \neq j$, then the x^i can slip inside the derivative. This would give us the integral of a total derivative, which vanishes. So, we can get a nonzero result if and only if $i = j$. Thus:

$$[K_i, P_j] = -i\delta_{ij}\left\{\frac{1}{2}\int d^3x x^i \nabla^i (\Pi)^2 + \frac{1}{2}\int d^3x x^i \nabla^i (\nabla^k \phi)^2 + \int d^3x x^i \nabla^i V(\phi)\right\}$$

Note that we must write the δ_{ij} rather than simply setting $i = j$, since i and j are "independent" variables, ie they are set on the left hand side of the equation. Now we can integrate by parts in all three terms, the result is:

$$[K_i, P_j] = i\delta_{ij}\left\{\frac{1}{2}\int d^3x (\Pi)^2 + \frac{1}{2}\int d^3x (\nabla^k \phi)^2 + \int d^3x V(\phi)\right\}$$

which is:

$$[K_i, P_j] = i\delta_{ij}\int d^3x \left\{\frac{1}{2}(\Pi)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi)\right\}$$

Hence,

$$\boxed{[K_i, P_j] = i\delta_{ij} H}$$

which completes our verification of equation 2.19.

Next:

$$[J_i, J_j] = \frac{1}{4}\varepsilon_{iab}\varepsilon_{jcd}\left[M^{ab}, M^{cd}\right]$$

$$\implies [J_i, J_j] = \frac{1}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x d^3y \left[\mathcal{M}^{0ab}(x), \mathcal{M}^{0cd}(y)\right]$$

$$\implies [J_i, J_j] = \frac{1}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x d^3y \left[x^a T^{0b}(x) - x^b T^{0a}(x), y^c T^{0d}(y) - y^d T^{0c}(y)\right]$$

$$\implies [J_i, J_j] = \frac{1}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x d^3y \left\{\left[x^a T^{0b}(x), y^c T^{0d}(y)\right] + (c \leftrightarrow d) + (a \leftrightarrow b) + (a, c \leftrightarrow b, d)\right\}$$

Note that the signs of the permutations are all positive due to the Levi-Cevita symbols. From now on we'll just write these last three terms as "perms" to save space. Then:

$$[J_i, J_j] = \frac{1}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x d^3y \left\{x^a y^c \left[T^{0b}(x), T^{0d}(y)\right] + (\text{perms})\right\}$$

Now we use our result from part (a):

$$[J_i, J_j] = \frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x d^3y \left\{x^a y^c \left[\Pi(x)\nabla_x^b \delta^3(x-y)\nabla^d\phi(y) - \Pi(y)\nabla_y^d\delta^3(x-y)\nabla^b\phi(x)\right] + (\text{perms})\right\}$$

Next we integrate by parts (to isolate the delta function), then use the delta function:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^c \nabla^b \left(x^a \Pi\right)\nabla^d\phi - x^a\nabla^d\left(x^c\Pi\right)\nabla^b\phi + (\text{perms})\right\}$$

The Levi-Cevita symbols require that $a \neq b$ and $c \neq d$, so:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^a x^c \nabla^b\Pi\nabla^d\phi - x^a x^c\nabla^d\Pi\nabla^b\phi + (\text{perms})\right\}$$

Now we integrate by parts in the second term:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^a x^c \nabla^b\Pi\nabla^d\phi + \Pi\nabla^d\left(x^a x^c\nabla^b\phi\right) + (\text{perms})\right\}$$

Since $c \neq d$, we remove that term from the derivative:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^a x^c \nabla^b\Pi\nabla^d\phi + \Pi x^c\nabla^d\left(x^a\nabla^b\phi\right) + (\text{perms})\right\}$$

Using the product rule:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^a x^c \nabla^b\Pi\nabla^d\phi + \Pi x^c x^a\nabla^d\nabla^b\phi + \Pi x^c\delta^{ad}\nabla^b\phi + (\text{perms})\right\}$$

Now we reverse the order of the derivatives in the second term:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \left\{x^a x^c \nabla^b\Pi\nabla^d\phi + \Pi x^c x^a\nabla^b\nabla^d\phi + \Pi x^c\delta^{ad}\nabla^b\phi + (\text{perms})\right\}$$

and integrate by parts in the second term, remembering that $a \neq b$:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x\left\{x^a x^c \nabla^b \Pi \nabla^d \phi - \nabla^b(\Pi x^c) x^a \nabla^d \phi + \Pi x^c \delta^{ad} \nabla^b \phi + (\text{perms})\right\}$$

Using the product rule:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x\left\{x^a x^c \nabla^b \Pi \nabla^d \phi - \Pi \delta^{bc} x^a \nabla^d \phi - x^a x^c \nabla^b \Pi \nabla^d \phi + \Pi x^c \delta^{ad} \nabla^b \phi + (\text{perms})\right\}$$

The first and third terms vanish:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x\left\{\Pi x^c \delta^{ad} \nabla^b \phi - \Pi \delta^{bc} x^a \nabla^d \phi + (\text{perms})\right\}$$

Writing these terms separately:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \Pi x^c \delta^{ad} \nabla^b \phi + \frac{i}{4}\varepsilon_{iab}\varepsilon_{jcd}\int d^3x \Pi \delta^{bc} x^a \nabla^d \phi + (\text{perms})$$

Now we use the delta functions:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{iab}\varepsilon_{jca}\int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4}\varepsilon_{iab}\varepsilon_{jbd}\int d^3x \Pi x^a \nabla^d \phi + (\text{perms})$$

Rewriting the Levi-Cevita symbols:

$$[J_i, J_j] = -\frac{i}{4}\varepsilon_{aib}\varepsilon_{acj}\int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4}\varepsilon_{bai}\varepsilon_{bjd}\int d^3x \Pi x^a \nabla^d \phi + (\text{perms})$$

Now we use the identity for Levi-Cevita symbols:

$$[J_i, J_j] = -\frac{i}{4}\left(\delta_{ic}\delta_{bj} - \delta_{ij}\delta_{bc}\right)\int d^3x \Pi x^c \nabla^b \phi + \frac{i}{4}\left(\delta_{aj}\delta_{id} - \delta_{ad}\delta_{ij}\right)\int d^3x \Pi x^a \nabla^d \phi + (\text{perms})$$

Expanding this:

$$[J_i, J_j] = \frac{i}{4}\left\{-\int d^3x \Pi x^i \nabla^j \phi + \int d^3x \Pi x^b \nabla^b \phi + \int d^3x \Pi x^j \nabla^i \phi - \int d^3x \Pi x^a \nabla^a \phi + (\text{perms})\right\}$$

In the second and fourth terms, a and b are reduced to dummy variables, and these terms cancel. Then:

$$[J_i, J_j] = \frac{i}{4}\left\{-\int d^3x \Pi x^i \nabla^j \phi + \int d^3x \Pi x^j \nabla^i \phi + (\text{perms})\right\}$$

Remember that our permutations involve permutating a, b, c, and d. But none of these terms matter anymore! So, the three permutations are identical to the first term, and:

$$[J_i, J_j] = i\left\{\int d^3x\left(\Pi x^j \nabla^i \phi - \Pi x^i \nabla^j \phi\right)\right\}$$

This, of course, is $\mathcal{M}$:

$$[J_i, J_j] = -i \left\{ \int d^3x \mathcal{M}^{0ji} \right\}$$

which is:

$$[J_i, J_j] = -iM^{ji}$$

Since M is anti-symmetric:

$$[J_i, J_j] = iM^{ij}$$

Now let's write this differently:

$$[J_i, J_j] = \frac{i}{2}(M^{ij} - M^{ji})$$

$$\implies [J_i, J_j] = \frac{i}{2}(\delta_{jb}\delta_{ia} - \delta_{ja}\delta_{ib})M^{ab}$$

$$\implies [J_i, J_j] = \frac{i}{2}\varepsilon_{kij}\varepsilon_{kab}M^{ab}$$

$$\implies [J_i, J_j] = \frac{i}{2}\varepsilon_{ijk}\varepsilon_{kab}M^{ab}$$

which implies:

$$\boxed{[J_i, J_j] = i\varepsilon_{ijk}J_k}$$

Next:

$$[J_i, K_j] = \left[\frac{1}{2}\varepsilon_{iab}M^{ab}, M^{j0}\right]$$

$$\implies [J_i, K_j] = \frac{1}{2}\varepsilon_{iab}\left[M^{ab}, M^{j0}\right]$$

$$\implies [J_i, K_j] = \frac{1}{2}\varepsilon_{iab}\int d^3x d^3y \left[\mathcal{M}^{0ab}(x), \mathcal{M}^{0j0}(y)\right]$$

$$\implies [J_i, K_j] = \frac{1}{2}\varepsilon_{iab}\int d^3x d^3y \left[x^a T^{0b}(x) - x^b T^{0a}(x), x^j T^{00}(y) - x^0 T^{0j}(y)\right]$$

Note that $x^0 = t$. Let's choose to evaluate the commutator at $t = 0$. Since commutators should not be time-dependent, the value at $t = 0$ should equal the commutator at all other times. Hence,

$$[J_i, K_j] = \frac{1}{2}\varepsilon_{iab}\int d^3x d^3y \left\{x^a x^j \left[T^{0b}(x), T^{00}(y)\right] + (a \leftrightarrow b)\right\}$$

Using our result from part (a):

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x d^3y \Big\{x^a x^j \nabla_x^k\phi(x)\nabla_x^k\phi(x)\nabla_x^k\delta^3(x-y)\nabla_y^b\phi(y) + x^a x^j \frac{\delta V}{\delta\phi}\delta^3(x-y)\nabla_y^b\phi(y)$$

$$-x^a x^j \Pi(y)\Pi(x)\nabla_y^b\delta^3(x-y) + (a \leftrightarrow b)\Big\}$$

Integrating by parts:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x d^3y \left\{ -\nabla^b_y\phi(y)\delta^3(x-y)\nabla^k_x\left[x^a x^j \nabla^k_x\phi(x)\right] + x^a x^j \frac{\delta V}{\delta\phi}\delta^3(x-y)\nabla^b_y\phi(y) \right.$$
$$\left. + x^a x^j \Pi(x)\nabla^b_y\Pi(y)\delta^3(x-y) + (a\leftrightarrow b)\right\}$$

Starting now we'll supress the permutation until the end of the problem. Using the delta functions:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -\nabla^b\phi\nabla^k\left[x^a x^j \nabla^k\phi\right] + x^a x^j \frac{\delta V}{\delta\phi}\nabla^b\phi + x^a x^j \Pi\nabla^b\Pi\right\}$$

Now in this first term, I want to integrate by parts, reverse the order of the derivatives, then integrate by parts again. This gives:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -\nabla^k\phi\nabla^b\left[x^a x^j \nabla^k\phi\right] + x^a x^j \frac{\delta V}{\delta\phi}\nabla^b\phi + x^a x^j \Pi\nabla^b\Pi\right\}$$

If $b \neq k$, then this first term is the integral of a total derivative, which vanishes. So, this is nonzero only if b = k, so:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -\nabla^b\phi\nabla^b\left[x^a x^j \nabla^b\phi\right] + x^a x^j \frac{\delta V}{\delta\phi}\nabla^b\phi + x^a x^j \Pi\nabla^b\Pi\right\}$$

The Levi-Cevita symbol requires that $a \neq b$, so:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -x^a\nabla^b\phi\nabla^b\left[x^j \nabla^b\phi\right] + x^a x^j \frac{\delta V}{\delta\phi}\nabla^b\phi + x^a x^j \Pi\nabla^b\Pi\right\}$$

Using the product rule:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -x^a\nabla^b\phi\delta^{bj}\nabla^b\phi - x^a x^j\nabla^b\phi\nabla^b\nabla^b\phi + x^a x^j \frac{\delta V}{\delta\phi}\nabla^b\phi + x^a x^j \Pi\nabla^b\Pi\right\}$$

The second, third, and fourth terms can be rewritten:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -x^a\nabla^b\phi\delta^{bj}\nabla^b\phi - \frac{1}{2}x^a x^j\nabla^b(\nabla^b\phi)^2 + x^a x^j\nabla^b V(\phi) + \frac{1}{2}x^a x^j\nabla^b(\Pi)^2\right\}$$

Now $a \neq b$, so:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -x^a\nabla^b\phi\delta^{bj}\nabla^b\phi - \frac{1}{2}x^j\nabla^b\left[x^a(\nabla^b\phi)^2\right] + x^a x^j\nabla^b V(\phi) + \frac{1}{2}x^a x^j\nabla^b(\Pi)^2\right\}$$

The second term vanishes if $b \neq j$. So:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab}\int d^3x \left\{ -x^a\nabla^b\phi\delta^{bj}\nabla^b\phi - \frac{1}{2}\delta^{bj}x^b\nabla^b\left[x^a(\nabla^b\phi)^2\right] + x^a x^j\nabla^b V(\phi) + \frac{1}{2}x^a x^j\nabla^b(\Pi)^2\right\}$$

Now let's integrate by parts in the second, third, and fourth terms, recalling that $a \neq b$:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab} \int d^3x \left\{ -x^a(\nabla\phi)^2\delta^{bj} + \frac{1}{2}\delta^{bj}x^a(\nabla\phi)^2 - x^a\delta^{bj}V(\phi) - \frac{1}{2}x^a\delta^{bj}(\Pi)^2 \right\}$$

Now these first two terms combine:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iab} \int d^3x \left\{ -\frac{1}{2}x^a(\nabla\phi)^2\delta^{bj} - x^a\delta^{bj}V(\phi) - \frac{1}{2}x^a\delta^{bj}(\Pi)^2 \right\}$$

Using the deltas:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{iaj} \int d^3x \left\{ -\frac{1}{2}x^a(\Pi)^2 - \frac{1}{2}x^a(\nabla\phi)^2 - x^aV(\phi) \right\}$$

Reversing the Levi-Cevita symbol, and re-inserting the permutations:

$$[J_i, K_j] = \frac{i}{2}\varepsilon_{ija} \int d^3x \left\{ \frac{1}{2}x^a(\Pi)^2 + \frac{1}{2}x^a(\nabla\phi)^2 + x^aV(\phi) + (a \leftrightarrow b) \right\}$$

Now b is gone and a is just a dummy variable, so the permutations is equal to the original term:

$$[J_i, K_j] = i\varepsilon_{ija} \int d^3x \left\{ \frac{1}{2}x^a(\Pi)^2 + \frac{1}{2}x^a(\nabla\phi)^2 + x^aV(\phi) \right\}$$

Let's also rename the dummy variable, $a \to k$:

$$[J_i, K_j] = i\varepsilon_{ijk} \int d^3x \left\{ \frac{1}{2}x^k(\Pi)^2 + \frac{1}{2}x^k(\nabla\phi)^2 + x^kV(\phi) \right\}$$

This obviously looks like the Hamiltonian density, so:

$$[J_i, K_j] = i\varepsilon_{ijk} \int d^3x [x^k T^{00}(x)]$$

Remembering that this should be time-independent, we know that any time-dependent term should vanish. So:

$$[J_i, K_j] = i\varepsilon_{ijk} \int d^3x [x^k T^{00}(x) - x^0 T^{0k}(x)]$$

which is equal to:

$$[J_i, K_j] = i\varepsilon_{ijk} \int d^3x \mathcal{M}^{0k0}(x)$$

which is:

$$[J_i, K_j] = i\varepsilon_{ijk} M^{k0}$$

and so:

$$\boxed{[J_i, K_j] = i\varepsilon_{ijk} K_k}$$

Finally:

$$[K_i, K_j] = \left[M^{i0}, M^{j0}\right]$$

$$\implies [K_i, K_j] = \int d^3x d^3y \left[\mathcal{M}^{0i0}, \mathcal{M}^{0j0}\right]$$

$$\implies [K_i, K_j] = \int d^3x d^3y \left[x^i T^{00}(x), y^j T^{00}(y)\right]$$

where, as before, we neglected those terms with explicit time-dependence, choosing to work where $t = 0$.

$$\implies [K_i, K_j] = \int d^3x d^3y x^i y^j \left[T^{00}(x), T^{00}(y)\right]$$

Now this looks very familiar to our expression for $[K_i, H]$. In fact, we can insert the x^j into equation (22.3.1), renaming the dummy index:

$$[K_i, K_j] = i \int d^3x \left\{\nabla^k(x^j \nabla^k \phi)\Pi x^i - \nabla^k(x^i \nabla^k \phi)\Pi x^j\right\}$$

Using the product rule:

$$[K_i, K_j] = i \int d^3x \left\{\nabla^k \phi \delta^{jk} \Pi x^i + x^j \nabla^2 \phi \Pi x^i - \nabla^k \phi \delta^{ik} \Pi x^j - x^i \nabla^2 \phi \Pi x^j\right\}$$

The second and fourth terms vanish:

$$[K_i, K_j] = i \int d^3x \left\{\nabla^k \phi \delta^{jk} \Pi x^i - \nabla^k \phi \delta^{ik} \Pi x^j\right\}$$

Using the deltas:

$$\implies [K_i, K_j] = i \int d^3x \left\{\nabla^j \phi \Pi x^i - \nabla^i \phi \Pi x^j\right\}$$

$$\implies [K_i, K_j] = i \int d^3x \left\{-T^{0j} x^i + T^{0i} x^j\right\}$$

$$\implies [K_i, K_j] = -i \int d^3x \left\{T^{0j} x^i - T^{0i} x^j\right\}$$

$$\implies [K_i, K_j] = -i \int d^3x \mathcal{M}^{0ij}$$

$$\implies [K_i, K_j] = -i M^{ij}$$

Now let's write this differently

$$[K_i, K_j] = -\frac{i}{2}(M^{ij} - M^{ji})$$

$$\implies [K_i, K_j] = -\frac{i}{2}(\delta_{jb}\delta_{ia} - \delta_{ja}\delta_{ib}) M^{ab}$$

$$\implies [K_i, K_j] = -\frac{i}{2}\varepsilon_{kij}\varepsilon_{kab} M^{ab}$$

$$\implies [K_i, K_j] = -\frac{i}{2}\varepsilon_{ijk}\varepsilon_{kab} M^{ab}$$

which implies:

$$\boxed{[K_i, K_j] = -i\varepsilon_{ijk} J_k}$$

which completes our verification of equation 2.17.