Srednicki Chapter 21

QFT Problems & Solutions

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Srednicki 21.1. Show that

$$\boldsymbol{\Gamma(\phi) = W(J_\phi) - \int d^d x J_\phi \phi}$$

where $\mathbf{J}_\phi$ is the solution of

$$\boldsymbol{\frac{\delta}{\delta J(x)} W(J) = \phi(x)}$$

for a specified $\boldsymbol{\phi}(\mathbf{x})$.

By comparing equations 21.14 and 21.18, we find that:

$$\frac{\delta}{\delta J(x)} W(J) = \phi(x) \tag{21.1.1}$$

Meanwhile, equation 21.12 gives:

$$W(J) = \Gamma(\phi_J) + \int d^d x J \phi_J$$

This is true for all J. Now let's choose the J that satisfies equation 21.21:

$$W(J_\phi) = \Gamma(\phi_J) + \int d^d x J_\phi \phi_J$$

Comparing equation (21.1.1) with equation 21.21, we find that $\phi = \phi_J$. Therefore,

$$W(J_\phi) = \Gamma(\phi) + \int d^d x J_\phi \phi$$

which gives:

$$\Gamma(\phi) = W(J_\phi) - \int d^d x J_\phi \phi$$

which is equation 21.20.

Srednicki 21.2 *Symmetries of the quantum action.* Suppose that we have a set of fields $\phi_a(x)$ and that both the classical action $S(\phi)$ and the integration measure $\mathcal{D}\phi$ are invariant under

$$\phi_a \to \int d^d y R_{ab}(x,y)\phi_b(y)$$

for some particular function $R_{ab}(x,y)$. Typically $R_{ab}(x,y)$ is a constant matrix times $\delta^d(x-y)$, or a finite number of derivatives of $\delta^d(x-y)$; see section 22-24 for some examples.

(a) Show that W(J) is invariant under

$$J_a(x) \to \int d^d y J_b(y) R_{ba}(y,x)$$

Equation 21.3 implies that $W(J)$ is invariant if Z(J) is. Equation 21.1 shows that:

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x J\phi\right]$$

This is true for all ϕ. Let's decide that the ϕ in question is $\phi' = \int d^d y R_{ab}(x,y)\phi_b$. Notice that this is not necessarily a transformation, just a change of the dummy integration variable, which is always allowed. Then,

$$Z(J) = \int \mathcal{D}\phi' \exp\left[iS(\phi') + i\int d^d x J\phi'\right]$$

The problem statement tells us that the integration measure and the classical action are invariant. Hence:

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x J\phi'\right]$$

Using the definition of ϕ' in the remaining term gives:

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x d^d y J_a R_{ab}(x,y)\phi_b\right] \tag{21.2.1}$$

where the J assumes a subscript to indicate that it is still to be multiplied by the ϕ'.

Now let's use the transformation of J as written in equation 21.3:

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x d^d y J_b R_{ba}(y,x)\phi_a\right]$$

The dummy indices and the dummy variables can be reversed. So,

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x d^d y J_a R_{ab}(x,y)\phi_b\right] \tag{21.2.2}$$

Note that equation (21.2.1) and (21.2.2) are equal: thus, the transformation of J leaves Z(J) – and hence W(J) – invariant.

(b) Use eqs. 21.20 and 21.23 to show that the quantum action $\Gamma(\phi)$ is invariant under equation 21.22. This is an important result that we will use frequently.

Equation 21.20 states:

$$\Gamma(\phi) = W(J_\phi) - \int d^d x J_\phi \phi$$

Transforming $\Gamma(\phi)$ as indicated, we have:

$$\Gamma(\phi) \to W(J'_\phi) - \int d^d x d^d y J'_\phi R_{ab}(x,y)\phi_b(y)$$

Now J_ϕ depends on ϕ as well. Now let's impose the transformation $J_\phi \to \int d^d z J_\phi(z) R_{ab}(x,z)$. Otherwise we can't get back to equation 21.20. Using this assumption, we have:

$$\Gamma(\phi) \to W(J'_\phi) - \int d^d x d^d y d^d z J_\phi (R^{-1})_{ab}(x,y)\delta(x-y)R_{ab}(x,y)\phi_b(y)$$

At this point we have to assume that the R matrix has a delta function in it at some level. Srednicki stated that this is typically true, but we see that it must be absolutely true for this statement to hold. Then the R functions cancel, and we're left with:

$$\Gamma(\phi) \to W(J'_\phi) - \int d^d x J_\phi \phi$$

Finally, our result from part (a) tells us that W is invariant under this transformation. Thus,

$$\Gamma(\phi) \to W(J_\phi) - \int d^d x J_\phi \phi$$

But now we have to check our assumption, which is equivalent to defining $J'_\phi = \int d^d z J_\phi(z)(R^{-1})_{ab}(x,z)$. Since we assumed that all the R's have a delta function, the integrals cease to exist. Then, our assumption is good only if

$$\frac{\delta W(J)}{\delta J'(x)} \stackrel{?}{=} R_{ab}\phi_b$$

which implies

$$\frac{\delta W(J)}{\delta J(x)} \frac{\delta J(x)}{\delta J'(x)} \stackrel{?}{=} R_{ab}\phi_b$$

The first of these is given by equation 21.22; the second of these is given by our definion:

$$R_{ab}\phi_b \stackrel{\checkmark}{=} R_{ab}\phi_b$$

which is obviously true.

Note: Our assumption that the R's have a delta function is of paramount importance. If this delta function is missing, the entire argument is potentially invalid. Alternatively, the derivation would work if the integrals in equations 21.22 and 21.23 were missing (that's what Srednicki proves in his solutions).

Srednicki 21.3. Consider performing the path integral in the presence of a background field $\bar{\phi}(x)$; we define

$$\boldsymbol{exp[iW(J;\bar{\phi})] = \int \mathcal{D}\phi \; exp\left[iS(\phi+\bar{\phi}) + i\int d^dxJ\phi\right]}$$

Then W(J;0) is the original W(J) of equation 21.3. We also define the quantum action in the presence of the background field,

$$\boldsymbol{\Gamma(\phi;\bar{\phi}) = W(J_\phi;\bar{\phi}) - \int d^dxJ_\phi\phi}$$

where $J_\phi(x)$ is the solution of

$$\boldsymbol{\frac{\delta}{\delta J(x)}W(J;\bar{\phi}) = \phi(x)}$$

for a specified $\phi(x)$. Show that

$$\boldsymbol{\Gamma(\phi;\bar{\phi}) = \Gamma(\phi+\bar{\phi};0)}$$

where $\Gamma(\phi;0)$ is the original quantum action of equation 21.1.

Proceeding directly:

$$\Gamma(\phi+\bar{\phi};0) = W(J_{\phi+\bar{\phi}};0) - \int d^dxJ_{\phi+\bar{\phi}}(\phi+\bar{\phi}) \tag{21.3.1}$$

Stuck already. To proceed, we have to know what $J_{\phi+\bar{\phi}}$ is. Using the analog of equation 21.26, we see that $J_{\phi+\bar{\phi}}$ is the solution of

$$\frac{\delta}{\delta J(x)}W(J_{\phi+\bar{\phi}};0) = \phi(x) + \bar{\phi}(x) \tag{21.3.2}$$

Again, to proceed we have to know what $W(J_{\phi+\bar{\phi}};0)$ is. To determine this, we go to equation 21.24:

$$exp[iW(J;\bar{\phi})] = \int \mathcal{D}\phi \; exp\left[iS(\phi+\bar{\phi}) + i\int d^dxJ\phi\right]$$

Switching our integration variable to $\phi' = \phi + \bar{\phi}$, we have:

$$exp[iW(J;\bar{\phi})] = \int \mathcal{D}\phi' \; exp\left[iS(\phi') + i\int d^dxJ(\phi'-\bar{\phi})\right]$$

which is:

$$exp[iW(J;\bar{\phi})] = e^{-i\int d^dxJ\bar{\phi}}\int \mathcal{D}\phi' \; exp\left[iS(\phi') + i\int d^dxJ\phi'\right]$$

which gives:

$$exp[iW(J;\bar{\phi})] = exp\left[iW(J;0) - i\int d^dxJ\bar{\phi}(x)\right]$$

Simplifying this, we have:

$$W(J;\bar{\phi}) = W(J;0) - \int d^dxJ\bar{\phi}(x) \tag{21.3.3}$$

Next let's take the functional derivative of this with respect to J, and use equation 21.26. We find:

$$\phi(x) = \frac{\delta W(J;0)}{\delta J} - \bar{\phi}(x)$$

and so:

$$\frac{\delta W(J;0)}{\delta J} = \phi(x) + \bar{\phi}(x)$$

Comparing this to equation (21.3.2), we see that $J_\phi = J_{\phi+\bar{\phi}}$. With this, we can go back to equation (21.3.1):

$$\Gamma(\phi+\bar{\phi};0) = W(J_\phi;0) - \int d^dxJ_\phi\phi - \int d^dxJ_\phi\bar{\phi}$$

Using equation (21.3.3):

$$\Gamma(\phi+\bar{\phi};0) = W(J;\bar{\phi}) - \int d^dxJ_\phi\phi$$

which is, from equation 21.25:

$$\Gamma(\phi+\bar{\phi};0) = \Gamma(\phi;\bar{\phi})$$

as expected.