# Srednicki Chapter 20

QFT Problems & Solutions

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**Srednicki 20.1. Verify equation 20.17.**

Using equation 20.7, 20.11, and the fact that $m = 0$ in this limit, our task is to evaluate this integral:

$$3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \frac{1}{x_2(-sx_1 + tx_3) + (-x_3 t + x_1 x_3 t + x_3^2 t)}$$

These are all dummy indices, so let's swap $x_2 \leftrightarrow x_3$ in the integrand:

$$3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \frac{1}{x_3(-sx_1 + tx_2) + (-x_2 t + x_1 x_2 t + x_2^2 t)}$$

It is trivial that:

$$\int \frac{dx}{Ax + B} = \frac{1}{A} ln(Ax + B)$$

So we solve this integral to find:

$$3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left\{ \ln \left[ (1 - x_1 - x_2)(-sx_1 + tx_2) + (-x_2 t + x_1 x_2 t + x_2^2 t) \right] \right.$$

$$\left. - \ln \left[ (-x_2 t + x_1 x_2 t + x_2^2 t) \right] \right\}$$

Using properties of the logarithm to simplify this, we find that:

$$3! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\ln s + \ln x_1 - \ln t - \ln x_2}{tx_2 - sx_1}$$

If we let $u = tx_2 - sx_1$, we have:

$$\frac{3!}{t} \int_0^1 dx_1 \int_{x_2=0}^{x_2=1-x_1} du \left\{ \frac{\ln\left(\frac{sx_1}{t}\right)}{u} - \frac{\ln\left(\frac{u+sx_1}{t}\right)}{u} \right\}$$

Let's notice that the denominators in both logarithms cancel (using properties of the logarithm). Then,

$$\frac{3!}{t} \int_0^1 dx_1 \int_{x_2=0}^{x_2=1-x_1} du \left\{ \frac{\ln(sx_1)}{u} - \frac{\ln(u + sx_1)}{u} \right\}$$

This first integral is easy to evaluate:

$$\frac{3!}{t}\int_0^1 dx_1\left\{\ln(sx_1)\ln\left(\frac{t+s}{s}-\frac{t}{sx_1}\right)-\int_{x_2=0}^{x_2=1-x_1}du\frac{\ln(u+sx_1)}{u}\right\} \tag{20.1.1}$$

The second integral needs to be evaluated on Mathematica and simplified by hand; we find that:

$$\int du\frac{\ln(u+sx_1)}{u}=\ln(u)\ln(sx_1)-\text{Polylog}_2\left(-\frac{u}{sx_1}\right)$$

Now let's go back to $x_2$ so we can apply the endpoints:

$$\int du\frac{\ln(u+sx_1)}{u}=\ln(tx_2-sx_1)\ln(sx_1)-\text{Polylog}_2\left(-\frac{tx_2-sx_1}{sx_1}\right)$$

And applying the endpoints:

$$\int_{x_2=0}^{x_2=1-x_1}du\frac{\ln(u+sx_1)}{u}=\ln(t(1-x_1)-sx_1)\ln(sx_1)-\text{Polylog}_2\left(-\frac{t(1-x_1)-sx_1}{sx_1}\right)$$
$$-\ln(-sx_1)\ln(sx_1)+\text{Polylog}_2\left(-\frac{-sx_1}{sx_1}\right)$$

Simplifying, and using the definition of the polylogarithm in the last term, we find:

$$\int_{x_2=0}^{x_2=1-x_1}du\frac{\ln(u+sx_1)}{u}=\ln\left(\frac{t(1-x_1)-sx_1}{-sx_1}\right)\ln(sx_1)-\text{Polylog}_2\left(-\frac{t(1-x_1)-sx_1}{sx_1}\right)+\frac{\pi^2}{6}$$

which is:

$$\int_{x_2=0}^{x_2=1-x_1}du\frac{\ln(u+sx_1)}{u}=\ln\left(\frac{t+s}{s}-\frac{t}{sx_1}\right)\ln(sx_1)-\text{Polylog}_2\left(\frac{t+s}{s}-\frac{t}{sx_1}\right)+\frac{\pi^2}{6}$$

Using this in equation (20.1.1):

$$\frac{3!}{t}\int_0^1 dx_1\text{Polylog}_2\left(\frac{t+s}{s}-\frac{t}{sx_1}\right)-\frac{\pi^2}{t} \tag{20.1.2}$$

Now let's redefine some things for simplicity. We'll take $x_1\to x$, $A=1+\frac{t}{s}$, and $B=\frac{t}{s}$. We'll also suppress the 3!/t, though it will be very important to add it in later. Finally, let's ignore the endpoints of the integral for now, defining the indefinite integral to be a function instead. Then,

$$f(x)=\int dx\text{Polylog}_2\left(A-\frac{B}{x}\right)$$

This can be evaluated on Mathematica:

$$f(x)=x\text{Polylog}_2\left(A-\frac{B}{x}\right)+\frac{B}{A}\left\{\ln\left(1-A+\frac{B}{x}\right)\ln\left(B-Ax\right)\right.$$
$$\left.+\ln\left(\frac{B}{1-A}+x\right)\left(-\ln\left(B-Ax\right)+\ln\left(\frac{(A-1)(Ax-B)}{B}\right)\right)\right.$$

$$+\ln(x)\left(\ln(B-Ax)-\ln\left(1-\frac{Ax}{B}\right)\right)-\text{Polylog}_2\left(\frac{Ax}{B}\right)+\text{Polylog}_2\left(\frac{A(B+x-Ax)}{B}\right)\Bigg\}$$

Using equation (20.1.2), our integral is equal to:

$$\frac{3!}{t}\left[f(1)-f(0)-\frac{\pi^2}{6}\right] \tag{20.1.3}$$

First let's evaluate f(1):

$$f(1)=\text{Polylog}_2\left(A-B\right)+\frac{B}{A}\{\ln\left(1-A+B\right)\ln\left(B-A\right)$$

$$+\ln\left(\frac{B}{1-A}+1\right)\left(-\ln\left(B-A\right)+\ln\left(\frac{(A-1)(A-B)}{B}\right)\right)$$

$$+\ln(1)\left(\ln(B-A)-\ln\left(1-\frac{A}{B}\right)\right)-\text{Polylog}_2\left(\frac{A}{B}\right)+\text{Polylog}_2\left(\frac{A(B+1-A)}{B}\right)\Bigg\}$$

It is easy to see (using properties of logs) that there are no infinities in the third-to-last term, so $\ln(1)$ sets the entire term to zero:

$$f(1)=\text{Polylog}_2\left(A-B\right)+\frac{B}{A}\{\ln\left(1-A+B\right)\ln\left(B-A\right)$$

$$+\ln\left(\frac{B-A+1}{1-A}\right)\left(-\ln\left(B-A\right)+\ln\left(\frac{(A-1)(A-B)}{B}\right)\right)$$

$$-\text{Polylog}_2\left(\frac{A}{B}\right)+\text{Polylog}_2\left(\frac{A(B+1-A)}{B}\right)\Bigg\}$$

Let's distribute the terms on the second line:

$$f(1)=\text{Polylog}_2\left(A-B\right)+\frac{B}{A}\{\ln\left(1-A+B\right)\ln\left(B-A\right)$$

$$-\ln\left(\frac{1-A+B}{1-A}\right)\ln\left(B-A\right)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{(A-1)(A-B)}{B}\right)$$

$$-\text{Polylog}_2\left(\frac{A}{B}\right)+\text{Polylog}_2\left(\frac{A(B+1-A)}{B}\right)\Bigg\}$$

Next, note that if we split the numerator from the denominator on the first term of the second line, there is a cancellation. Hence:

$$f(1)=\text{Polylog}_2\left(A-B\right)+\frac{B}{A}\Bigg\{\ln\left(1-A\right)\ln\left(B-A\right)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{(A-1)(A-B)}{B}\right)$$

$$-\text{Polylog}_2\left(\frac{A}{B}\right)+\text{Polylog}_2\left(\frac{A(B+1-A)}{B}\right)\Bigg\}$$

Now, note that $A-B=1$. This lets us simplify many terms:

$$f(1)=\text{Polylog}_2\left(1\right)+\frac{B}{A}\Bigg\{\ln\left(1-A\right)\ln\left(-1\right)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{A-1}{B}\right)$$

$$-\text{Polylog}_2\left(\frac{A}{B}\right)+\text{Polylog}_2\,(0)\Bigg\}$$

This last term vanishes:

$$f(1)=\text{Polylog}_2\,(1)+\frac{B}{A}\left\{\ln(1-A)\ln(-1)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{A-1}{B}\right)-\text{Polylog}_2\left(\frac{A}{B}\right)\right\}$$

Using the definition of the Polylog, we find the value of this first term:

$$f(1)=\frac{\pi^2}{6}+\frac{B}{A}\left\{\ln(1-A)\ln(-1)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{A-1}{B}\right)-\text{Polylog}_2\left(\frac{A}{B}\right)\right\}$$

Now let's go back to the definition of f(x) to calculate f(0):

$$f(x)=x\text{Polylog}_2\left(A-\frac{B}{x}\right)+\frac{B}{A}\Bigg\{\ln\left(1-A+\frac{B}{x}\right)\ln(B-Ax)$$

$$+\ln\left(\frac{B}{1-A}+x\right)\left(-\ln(B-Ax)+\ln\left(\frac{(A-1)(Ax-B)}{B}\right)\right)$$

$$+\ln(x)\left(\ln(B-Ax)-\ln\left(1-\frac{Ax}{B}\right)\right)-\text{Polylog}_2\left(\frac{Ax}{B}\right)+\text{Polylog}_2\left(\frac{A(B+x-Ax)}{B}\right)\Bigg\}$$

The first term vanishes as $x\to 0$. The easiest way to verify this is to try a few values on Mathematica, or check the graph. It also makes sense conceptually, since the polylog is in some sense a "heavy-duty logarithm", which will grow much slower than $x$ shrinks. Let's plug in 0 where possible on the other terms, leaving the terms that appear to diverge as x for the moment:

$$f(0)=\frac{B}{A}\Bigg\{\ln\left(1-A+\frac{B}{x}\right)\ln(B)+\ln\left(\frac{B}{1-A}\right)(-\ln(B)+\ln(1-A))$$

$$+\ln(x)\left(\ln(B)-\ln(1)\right)-\text{Polylog}_2\,(0)+\text{Polylog}_2\,(A)\}$$

We know that $\text{Polylog}_2(0)=0$. We can also combine the two added logarithms on the first line. This gives:

$$f(0)=\frac{B}{A}\Bigg\{\ln\left(1-A+\frac{B}{x}\right)\ln(B)+\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)$$

$$+\ln(x)\ln(B)+\text{Polylog}_2\,(A)\}$$

The third and first terms can now be combined, using the properties of logarithms:

$$f(0)=\frac{B}{A}\left\{\ln(x-Ax+B)\ln(B)+\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)+\text{Polylog}_2\,(A)\right\}$$

Setting the last remaining xs to 0, we have:

$$f(0)=\frac{B}{A}\left\{\ln(B)\ln(B)+\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)+\text{Polylog}_2\,(A)\right\}$$

Now we're ready to combine these:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{\ln(1-A)\ln(-1)+\ln\left(\frac{B-A+1}{1-A}\right)\ln\left(\frac{A-1}{B}\right)-\mathrm{Polylog}_2\left(\frac{A}{B}\right)\right.$$
$$\left.-\ln(B)\ln(B)-\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)-\mathrm{Polylog}_2(A)\right\}$$

Let's separate the denominator in the second term inside the braces:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{\ln(1-A)\ln(-1)+\ln(B-A+1)\ln\left(\frac{A-1}{B}\right)\right.$$
$$-\ln(1-A)\ln\left(\frac{A-1}{B}\right)-\mathrm{Polylog}_2\left(\frac{A}{B}\right)-\ln(B)\ln(B)-\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)$$
$$-\mathrm{Polylog}_2(A)\}$$

Combining the first and third terms:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{-\ln(1-A)\ln\left(\frac{1-A}{B}\right)+\ln(B-A+1)\ln\left(\frac{A-1}{B}\right)\right.$$
$$\left.-\mathrm{Polylog}_2\left(\frac{A}{B}\right)-\ln(B)\ln(B)-\ln\left(\frac{B}{1-A}\right)\ln\left(\frac{1-A}{B}\right)-\mathrm{Polylog}_2(A)\right\}$$

Combining the first and fifth terms:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{-\ln(B)\ln\left(\frac{1-A}{B}\right)+\ln(B-A+1)\ln\left(\frac{A-1}{B}\right)\right.$$
$$\left.-\mathrm{Polylog}_2\left(\frac{A}{B}\right)-\ln(B)\ln(B)-\mathrm{Polylog}_2(A)\right\}$$

Combining the first and fourth terms:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{-\ln(B)\ln(1-A)+\ln(B-A+1)\ln\left(\frac{A-1}{B}\right)\right.$$
$$\left.-\mathrm{Polylog}_2\left(\frac{A}{B}\right)-\mathrm{Polylog}_2(A)\right\}$$

$1-A=-B$, so the first term becomes:

$$f(1)-f(0)=\frac{\pi^2}{6}+\frac{B}{A}\left\{-\ln(B)\ln(-B)+\ln(B-A+1)\ln\left(\frac{A-1}{B}\right)\right.$$
$$\left.-\mathrm{Polylog}_2\left(\frac{A}{B}\right)-\mathrm{Polylog}_2(A)\right\}$$

The second term has two components multiplied together: the first forces the function toward $-\infty$, the other toward 0. Using L'hôpital's rule (or another means), we see that this vanishes. Then,

$$f(1) - f(0) = \frac{\pi^2}{6} + \frac{B}{A}\left\{-\ln(B)\ln(-B) - \text{Polylog}_2\left(\frac{A}{B}\right) - \text{Polylog}_2(A)\right\}$$

Using this in (20.1.3), the value of the integral is:

$$\frac{6B}{At}\left\{-\ln(B)\ln(-B) - \text{Polylog}_2\left(\frac{A}{B}\right) - \text{Polylog}_2(A)\right\}$$

$B/A = t/(s+t)$, so:

$$-\frac{6}{s+t}\left\{\ln\left(\frac{t}{s}\right)\ln\left(-\frac{t}{s}\right) + \text{Polylog}_2\left(\frac{s+t}{t}\right) + \text{Polylog}_2\left(\frac{s+t}{s}\right)\right\}$$

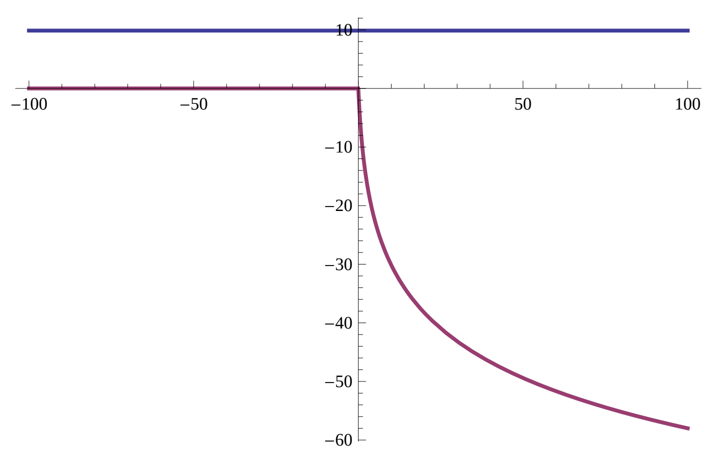
which implies:

$$-\frac{6}{s+t}\left\{\ln\left(\frac{s}{t}\right)^2 - i\pi\ln\left(\frac{s}{t}\right) + \text{Polylog}_2\left(\frac{s+t}{t}\right) + \text{Polylog}_2\left(\frac{s+t}{s}\right)\right\}$$

Now we need to use an identity, which holds when $x < 0$.

$$\pi^2 = -2i\pi\ln(x) + 2\text{PolyLog}_2(1+1/x) + 2\text{PolyLog}_2(1+x) + \ln(x)^2$$

I can't find a statement of this identity anywhere. Mathematically proving this identity ourselves would be very difficult, but fortunately we're not mathematicians. We're convinced if we just look at a graph of the right hand side:




where the blue is the real part and the purple is the imaginary part. Indeed, the identity appears to be true for $x < 0$. Using this in our expression, with $x = s/t$, we obtain equation 20.17:

$$\int \frac{dF_4}{D_4(s,t)} = -\frac{3}{s+t}\left(\pi^2 + [\ln(s/t)]^2\right)$$

*Note: Srednicki's solution presents an easier way to derive this. Curiously enough, by comparing his answer with my answer, we are able to mathematically prove the identity I stated above (since our answers are equivalent if and only if the identity is true). I presented this solution instead, because I think it is unlikely that the "trick" Srednicki used would be obvious to a student without further help. Of course, I have had to pay dearly for this rectitude – my solution is much more complicated than Srednicki's!*

*Note 2: Again, I'm unsure what the point of this problem was. Calculus practice? It is useful to practice Feynman's Formula – and the introduction to polylogarithms was certainly interesting – but this seems like a very difficult problem with very little purpose. Perhaps some leading hints could have at least allowed us to spend much less time on this problem.*

**Srednicki 20.2. Compute the O($\boldsymbol{\alpha}$) correction to the two-particle scattering amplitude at threshold, that is, for $\mathbf{s = 4m^2}$ and $\mathbf{t = u = 0}$, corresponding to zero three-momentum for both the incoming and outgoing particles.**

All we have to do is evaluate some unpleasant integrals. Let's start with $V_3(0)$:

$$V_3(0) = g - \frac{g\alpha 2!}{2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[1 - (x_1 + x_2)(1 - x_1 - x_2)\right]$$

We can rewrite this as:

$$V_3(0) = g - g\alpha \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[x_2^2 + (2x_1 - 1)x_2 + (1 - x_1 + x_1^2)\right]$$

This integral is an absolute mess, but after doing a lot of simplification, and using Mathematica, we find that:

$$V_3(0) = g - g\alpha \int_0^1 dx \left[-2 + 2x + \frac{\pi\sqrt{3}}{6} - \sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \left(x - \frac{1}{2}\right)\ln\left(1 - x_1 + x_1^2\right)\right]$$

Putting this on Mathematica, we find:

$$\boxed{V_3(0) = g + g\alpha\left(1 - \frac{\pi\sqrt{3}}{6}\right)}$$

Next we'll do $V_3(4m^2)$. Notice that the only difference is an extra term of $-4x_1x_2m^2$. Then,

$$V_3(4m^2) = g - g\alpha \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[x_2^2 + (-2x_1 - 1)x_2 + (1 - x_1 + x_1^2)\right]$$

Again, after integrating and doing a lot of simplification, we're left with:

$$V_3(4m^2) = g - g\alpha \int_0^1 dx \left[-2 + 2x + \sqrt{3 - 8x}\left\{\tan^{-1}\left(\frac{1+2x}{\sqrt{3-8x}}\right) - \tan^{-1}\left(\frac{4x-1}{\sqrt{3-8x}}\right)\right\}\right.$$

$$
+\frac{1-4x}{2}\ln\left(1-4x+4x^2\right)+\frac{1+2x}{2}\ln\left(1-x+x^2\right)\Bigg]
$$

Putting this on Mathematica, we find

$$
\boxed{V_3(4m^2)=g+g\alpha\left(\frac{8-\pi\sqrt{3}}{6}\right)}
$$

The four-point vertex function is given by:

$$
V_4(4m^2,0,0)=\frac{g^2\alpha}{m^2}\int_0^1 dx_1\int_0^{1-x_1}dx_2\int_0^{1-x_1-x_2}dx_3\left\{\frac{1}{-4x_1x_2+1-(x_1+x_2)(1-x_1-x_2)}\right.
$$

$$
\left.+\frac{1}{1-(x_1+x_2)(1-x_1-x_2)}+\frac{1}{-4x_3(1-x_1-x_2-x_3)+1-(x_1+x_2)(1-x_1-x_2)}\right\}
$$

It is best to do this on Mathematica; the result is:

$$
\boxed{V_4(4m^2,0,0)=\frac{g^2\alpha}{m^2}\left(-\frac{1}{3}-\frac{5\pi\sqrt{3}}{18}\right)}
$$

*Note: In an earlier version of this solution, I had a Mathematica workbook that replicated Srednicki's solution (though the answer did not seem intuitive). A year later, I received an e-mail from a reader who had typed in exactly the same thing, and gotten a different (but more intuitive) result to the first integral, resulting in a slightly different answer. I ended up reproducing his result, though I see no reason for the discrepancy. Maybe Mathematica was updated, and the bug was fixed? In any case, please see the attached workbook, updated, for a derivation of this answer – note that it does slightly disagree with Srednicki's solution. Thanks to Augusto Medeiros (Washington University in St Louis) for bringing this to my attention.*

The self-energy at zero is given by:

$$
\Pi(0)=-\frac{\alpha m^2}{2}\left\{\int_0^1 dx\ln\left(1-x+x^2\right)+\frac{1}{6}\right\}
$$

Using Mathematica:

$$
\Pi(0)=\frac{\alpha m^2}{2}\left[\frac{11}{6}-\frac{\pi}{\sqrt{3}}\right]
$$

which gives:

$$
\tilde{\mathbf{\Delta}}(0)=\frac{1}{m^2-\frac{\alpha m^2}{2}\left[\frac{11}{6}-\frac{\pi}{\sqrt{3}}\right]}
$$

Simplifying, and expanding the denominator to first order in $\alpha$, we have:

$$
\boxed{\tilde{\mathbf{\Delta}}(0)=\frac{1}{m^2}\left[1+\alpha\left(\frac{11-2\pi\sqrt{3}}{12}\right)\right]}
$$

Finally, the self-energy at $-4m^2$ is given by:

$$\Pi(-4m^2) = \frac{\alpha}{2}\int_0^1 dx(4x^2-4x+1)\ln\left(\frac{4x^2-4x+1}{x^2-x+1}\right) - \frac{\alpha}{12}3m^2$$

Doing the integral and simplifying, we find that:

$$\frac{\alpha m^2}{12}\left(2\sqrt{3}\pi - 9\right)$$

which gives:

$$\tilde{\mathbf{\Delta}}(-4m^2) = -\frac{1}{3m^2}\left(\frac{1}{1+\alpha\left[\frac{2\sqrt{3}\pi}{36} - \frac{1}{4}\right]}\right)$$

Expanding to first order in $\alpha$:

$$\boxed{\tilde{\mathbf{\Delta}}(-4m^2) = -\frac{1}{3m^2}\left(1+\alpha\left[\frac{1}{4} - \frac{2\sqrt{3}\pi}{36}\right]\right)}$$

Now we're ready to combine these using equation 20.2:

$$\tau_{1-loop} = -\frac{1}{3}\frac{g^2}{m^2} + \frac{g^2\alpha}{108m^2}(14\sqrt{3}\pi - 105) + 2\frac{g^2}{m^2} + g^2\alpha\frac{35-6\sqrt{3}\pi}{6m^2} + \frac{g^2\alpha}{m^2}\left(-\frac{1}{3} - \frac{5\pi\sqrt{3}}{18}\right)$$

Simplifying:

$$\boxed{\tau_{1-loop} = \tfrac{g^2}{108m^2}\left[180 + \alpha(489 - 124\sqrt{3}\pi)\right]}$$

As a fun fact, we can write this as:

$$\tau_{1-loop} = \frac{180g^2}{108m^2}\left[1 + \frac{\alpha(489-124\sqrt{3}\pi)}{180}\right] \approx \frac{180g^2}{108m^2}\left(1 - 0.000519g^2\right)$$

showing that these one-order corrections have an absolutely tiny impact on the overall scattering amplitude. In light of this, our decision to neglect all terms higher than $O(\alpha)$ seems extremely reasonable.

*Note: David Griffiths once wrote a problem prefaced by the comment "for masochists only." Srednicki would do well to add such a disclaimer to this problem.*

In[3]:=
```
Integrate[1 / (-4 * x1 * x2 + 1 - (x1 + x2) * (1 - x1 - x2)) + 1 / (1 - (x1 + x2) * (1 - x1 - x2)) +
  1 / (-4 * x3 * (1 - x1 - x2 - x3) + 1 - (x1 + x2) (1 - x1 - x2)), {x3, 0, 1 - x1 - x2}]
```

Out[3]=

$$\text{ConditionalExpression}\Big[-\frac{-1+x1+x2}{1+(-1+x1+x2)\,(x1+x2)}-\frac{-1+x1+x2}{1+x1^2+(-1+x2)\,x2-x1\,(1+2\,x2)}+\frac{\text{ArcTan}\left[\frac{1-x1-x2}{\sqrt{x1+x2}}\right]}{\sqrt{x1+x2}},$$

$$\left(\text{Im}\left[\frac{\sqrt{x1+x2}}{-1+x1+x2}\right]>1\;||\;\text{Im}\left[\frac{\sqrt{x1+x2}}{-1+x1+x2}\right]<-1\;||\;\text{Re}\left[\frac{\sqrt{x1+x2}}{-1+x1+x2}\right]\neq 0\right)\;\&\&$$

$$\left(\frac{-1+x1-\sqrt{-x1-x2}+x2}{-1+x1+x2}\notin \text{Reals}\;||\;\text{Re}\left[\frac{-1+x1-\sqrt{-x1-x2}+x2}{-1+x1+x2}\right]>2\;||\right.$$

$$\left.\text{Re}\left[\frac{-1+x1-\sqrt{-x1-x2}+x2}{-1+x1+x2}\right]<0\right)\;\&\&\;\left(\frac{-1+x1+\sqrt{-x1-x2}+x2}{-1+x1+x2}\notin \text{Reals}\;||\right.$$

$$\left.\text{Re}\left[\frac{-1+x1+\sqrt{-x1-x2}+x2}{-1+x1+x2}\right]>2\;||\;\text{Re}\left[\frac{-1+x1+\sqrt{-x1-x2}+x2}{-1+x1+x2}\right]<0\right)\Big]$$

In[14]:=
```
Integrate[-((-1 + x1 + x2) / (1 + (-1 + x1 + x2) (x1 + x2))) -
  (-1 + x1 + x2) / (1 + x1^2 + (-1 + x2) x2 - x1 (1 + 2 x2)) +
  ArcTan[(1 - x1 - x2) / Sqrt[x1 + x2]] / Sqrt[x1 + x2], x2]
```

Out[14]=

$$\sqrt{3}\;\text{ArcTan}\left[\frac{\sqrt{3}}{1-2\,x1-2\,x2}\right]-\frac{\text{ArcTan}\left[\frac{1-2\,x1-2\,x2}{\sqrt{3}}\right]}{\sqrt{3}}-\frac{\text{ArcTan}\left[\frac{1+2\,x1-2\,x2}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}+\frac{4\,x1\,\text{ArcTan}\left[\frac{1+2\,x1-2\,x2}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}+$$

$$2\,\sqrt{x1+x2}\;\text{ArcTan}\left[\frac{1-x1-x2}{\sqrt{x1+x2}}\right]-\frac{1}{2}\,\text{Log}\left[1-x1+x1^2-x2-2\,x1\,x2+x2^2\right]$$

In[16]:=
```
F[x1_, x2_] :=
 Sqrt[3] ArcTan[Sqrt[3] / (1 - 2 x1 - 2 x2)] - ArcTan[(1 - 2 x1 - 2 x2) / Sqrt[3]] / Sqrt[3] -
  ArcTan[(1 + 2 x1 - 2 x2) / Sqrt[3 - 8 x1]] / Sqrt[3 - 8 x1] +
  (4 x1 ArcTan[(1 + 2 x1 - 2 x2) / Sqrt[3 - 8 x1]]) / Sqrt[3 - 8 x1] +
  2 Sqrt[x1 + x2] ArcTan[(1 - x1 - x2) / Sqrt[x1 + x2]] -
  1 / 2 Log[1 - x1 + x1^2 - x2 - 2 x1 x2 + x2^2]
```

In[17]:= `F[x1, 1 - x1] - F[x1, 0]`

Out[17]=
$$-\sqrt{3}\,\text{ArcTan}\left[\frac{\sqrt{3}}{1-2\,x1}\right]+\frac{\text{ArcTan}\left[\frac{1-2\,x1}{\sqrt{3}}\right]}{\sqrt{3}}+\sqrt{3}\,\text{ArcTan}\left[\frac{\sqrt{3}}{1-2\,(1-x1)-2\,x1}\right]-$$
$$\frac{\text{ArcTan}\left[\frac{1-2\,(1-x1)-2\,x1}{\sqrt{3}}\right]}{\sqrt{3}}-2\,\sqrt{x1}\,\text{ArcTan}\left[\frac{1-x1}{\sqrt{x1}}\right]+\frac{\text{ArcTan}\left[\frac{1+2\,x1}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}-$$
$$\frac{4\,x1\,\text{ArcTan}\left[\frac{1+2\,x1}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}-\frac{\text{ArcTan}\left[\frac{1-2\,(1-x1)+2\,x1}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}+\frac{4\,x1\,\text{ArcTan}\left[\frac{1-2\,(1-x1)+2\,x1}{\sqrt{3-8\,x1}}\right]}{\sqrt{3-8\,x1}}+$$
$$\frac{1}{2}\,\text{Log}\left[1-x1+x1^2\right]-\frac{1}{2}\,\text{Log}\left[(1-x1)^2-2\,(1-x1)\,x1+x1^2\right]$$

In[18]:= `FullSimplify[%]`

Out[18]=
$$\frac{1}{18\,\sqrt{3-8\,x1}}$$
$$\left(-5\,\pi\,\sqrt{9-24\,x1}+18\,\sqrt{9-24\,x1}\,\text{ArcCot}\left[\frac{-1+2\,x1}{\sqrt{3}}\right]+36\,\sqrt{3-8\,x1}\,\sqrt{x1}\,\text{ArcTan}\left[\frac{-1+x1}{\sqrt{x1}}\right]-\right.$$
$$18\,(-1+4\,x1)\left(\text{ArcTan}\left[\frac{1+2\,x1}{\sqrt{3-8\,x1}}\right]-\text{ArcTan}\left[\frac{-1+4\,x1}{\sqrt{3-8\,x1}}\right]\right)-$$
$$\left.3\,\sqrt{3-8\,x1}\,\left(2\,\sqrt{3}\,\text{ArcTan}\left[\frac{-1+2\,x1}{\sqrt{3}}\right]+3\,\text{Log}\left[(1-2\,x1)^2\right]-3\,\text{Log}[1+(-1+x1)\,x1]\right)\right)$$

In[19]:=
```
Integrate[(1 / (18 Sqrt[3 - 8 x1]))
   (-5 π Sqrt[9 - 24 x1] + 18 Sqrt[9 - 24 x1] ArcCot[(-1 + 2 x1) / Sqrt[3]] +
     36 Sqrt[3 - 8 x1] Sqrt[x1] ArcTan[(-1 + x1) / Sqrt[x1]] - 18 (-1 + 4 x1)
      (ArcTan[(1 + 2 x1) / Sqrt[3 - 8 x1]] - ArcTan[(-1 + 4 x1) / Sqrt[3 - 8 x1]]) -
     3 Sqrt[3 - 8 x1] (2 Sqrt[3] ArcTan[(-1 + 2 x1) / Sqrt[3]] +
         3 Log[(1 - 2 x1) ^2] - 3 Log[1 + (-1 + x1) x1])), {x1, 0, 1}]
```

Out[19]=
$$\frac{1}{18}\left(-6-5\,\sqrt{3}\,\pi\right)$$