Srednicki Chapter 2

QFT Problems & Solutions

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Srednicki 2.1. Prove that $\delta\omega_{\rho\sigma}$ (the first-order term of the infinitesimal Lorentz matrix) is antisymmetric.

Start with the definition of the Lorentz Matrix:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

Use Srednicki 2.7, the infinitesmal form of the Lorentz Matrix,

$$g_{\mu\nu}(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \delta\omega^{\mu}{}_{\rho}) = g_{\rho\sigma}$$

Expanding, and throwing away the term with more than one differential:

$$g_{\mu\nu}\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + g_{\mu\nu}\delta\omega^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + g_{\mu\nu}\delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

Using these deltas:

$$g_{\rho\sigma} + g_{\mu\sigma}\delta\omega^{\mu}{}_{\rho} + g_{\rho\nu}\delta\omega^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

which gives:

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$$

Srednicki 2.2. Prove that 2.14 follows from $\mathbf{U}(\mathbf{\Lambda})^{-1}\mathbf{U}(\mathbf{\Lambda}')\mathbf{U}(\mathbf{\Lambda}) = \mathbf{U}(\mathbf{\Lambda}^{-1}\mathbf{\Lambda}'\mathbf{\Lambda})$

Starting from the assumption:

$$U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

Let $\Lambda' = 1 + \delta\omega'$.

$$U(\Lambda)^{-1}U(1+\delta\omega')U(\Lambda) = U(\Lambda^{-1}\Lambda + \Lambda^{-1}\delta\omega'\Lambda)$$

which implies:

$$U(\Lambda)^{-1}U(1+\delta\omega')U(\Lambda) = U(1 + \Lambda^{-1}\delta\omega'\Lambda)$$

Now we can use 2.12 on both sides (we'll drop the prime on the ω):

$$U(\Lambda)^{-1}(I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu})U(\Lambda) = I + \frac{i}{2\hbar}(\Lambda^{-1}\delta\omega\Lambda)_{\mu\nu}M^{\mu\nu}$$

The identity terms cancel, so do the constants. Further, we'll write the term on the right in index notation:

$$U(\Lambda)^{-1}(\delta\omega_{\mu\nu}M^{\mu\nu})U(\Lambda) = (\Lambda^{-1})_{\mu}{}^{\beta}\delta\omega_{\beta\sigma}(\Lambda)^{\sigma}{}_{\nu}M^{\mu\nu}$$

On the left, we note that $\delta\omega_{\mu\nu}$ is just a number, so it will commute with the operator. On the right, we'll use Srednicki 2.5 to deal with the inverse.

$$\delta\omega_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \delta\omega_{\beta\sigma}\Lambda^{\beta}{}_{\mu}(\Lambda)^{\sigma}{}_{\nu}M^{\mu\nu}$$

Now we'll use the argument in Srednicki's book: $\delta\omega$ is arbitrary. Therefore, its coefficients have to be the same at all times. But, the restriction is loosened a bit since M is antisymmetric. Thus, only the antisymmetric parts of the coefficients must be equal – the symmetric parts of the coefficients will go to zero when multiplied by M.

In this case, both the coefficients are completely antisymmetric, so we can equate them:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\beta}{}_{\mu}(\Lambda)^{\sigma}{}_{\nu}M^{\mu\nu}$$

as expected.

Srednicki 2.3. Verify that equation 2.16 follows from equation 2.14.

Equation 2.14 is:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}$$

Now let $\Lambda = 1 + \delta\omega$.

$$U(1-\delta\omega)M^{\mu\nu}U(1+\delta\omega) = (1+\delta\omega)^{\mu}{}_{\rho}(1+\delta\omega)^{\nu}{}_{\sigma}M^{\rho\sigma}$$

On the right hand side, we'll expand this and throw away anything higher than first order:

$$U(1-\delta\omega)M^{\mu\nu}U(1+\delta\omega) = M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu}$$

On the left hand side, we'll use 2.12:

$$\left(I - \frac{i}{2\hbar}\delta\omega_{\gamma\delta}M^{\gamma\delta}\right)M^{\mu\nu}\left(I + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) = M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu}$$

We again expand the left hand side and throw away anything higher than first order:

$$M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\gamma\delta}M^{\gamma\delta}M^{\mu\nu} + M^{\mu\nu}\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta} = M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu}$$

Cancel the first term on each side. Let's also use α and β universally for our dummy variables

$$-\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu} + M^{\mu\nu}\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta} = \delta\omega^{\nu}{}_{\alpha}M^{\mu\alpha} + \delta\omega^{\mu}{}_{\alpha}M^{\alpha\nu}$$

Finally, we'll insert some metrics as needed:

$$-\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu} + M^{\mu\nu}\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta} = g^{\nu\beta}\delta\omega_{\beta\alpha}M^{\mu\alpha} + g^{\mu\beta}\delta\omega_{\beta\alpha}M^{\alpha\nu}$$

In the third term, we'll reverse $\delta\omega$'s indices, remembering that it is antisymmetric. In the fourth term, we'll simply switch the indices (for free):

$$-\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu} + M^{\mu\nu}\frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta} = -g^{\nu\beta}\delta\omega_{\alpha\beta}M^{\mu\alpha} + g^{\mu\alpha}\delta\omega_{\alpha\beta}M^{\beta\nu}$$

Let's write some new terms on the right hand side. In these new terms, we switched the dummy indices (which is free), but then switched back $\delta\omega$, which cost a minus sign.

$$-\frac{i}{\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu}+M^{\mu\nu}\frac{i}{\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta} = -g^{\nu\beta}\delta\omega_{\alpha\beta}M^{\mu\alpha}+g^{\mu\alpha}\delta\omega_{\alpha\beta}M^{\beta\nu}+g^{\nu\alpha}\delta\omega_{\alpha\beta}M^{\mu\beta}-g^{\mu\beta}\delta\omega_{\alpha\beta}M^{\alpha\nu}$$

Now we remember that $\delta\omega$ is an arbitrary antisymmetric function, so the antisymmetric parts of its coefficients must be equal. In this case, the entire coefficient is antisymmetric, so we can equate them.

$$-\frac{i}{\hbar}M^{\alpha\beta}M^{\mu\nu} + M^{\mu\nu}\frac{i}{2\hbar}M^{\alpha\beta} = -g^{\nu\beta}M^{\mu\alpha} + g^{\mu\alpha}M^{\beta\nu} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\beta}M^{\alpha\nu}$$

This left hand side cleans up. We'll also multiply through by a minus sign:

$$\frac{1}{i\hbar}\left[M^{\mu\nu}, M^{\alpha\beta}\right] = g^{\nu\beta}M^{\mu\alpha} - g^{\mu\alpha}M^{\beta\nu} - g^{\nu\alpha}M^{\mu\beta} + g^{\mu\beta}M^{\alpha\nu}$$

Finally, we use the fact that g is symmetric and M is antisymmetric to get the form that we want:

$$\left[M^{\mu\nu}, M^{\alpha\beta}\right] = i\hbar\left(g^{\mu\alpha}M^{\nu\beta} - g^{\nu\alpha}M^{\mu\beta} + g^{\nu\beta}M^{\mu\alpha} - g^{\mu\beta}M^{\nu\alpha}\right)$$

which is equation 2.16.

Srednicki 2.4. Derive the commutation relations for P and J.

Using the definition of the angular momentum, $J_i = \frac{1}{2}\varepsilon^{ijk}M^{jk}$, hence:

$$J^1 = \frac{1}{2}M^{23} - \frac{1}{2}M^{32} = M^{23}$$

Therefore:

$$[J^1, J^2] = \left[M^{23}, M^{31}\right]$$
$$[J^1, J^2] = \left[M^{23}, M^{31}\right]$$

Using Srednicki 2.16 (which we proved in the previous problem):

$$[J^1, J^2] = i\hbar\left(g^{23}M^{31} - g^{33}M^{21} + g^{31}M^{23} - g^{21}M^{33}\right)$$

Using the metric:

$$[J^1, J^2] = -i\hbar M^{21}$$
$$[J^1, J^2] = i\hbar M^{12}$$

Hence:

$$[J^1, J^2] = i\hbar J^3$$
$$[J^2, J^1] = -i\hbar J^3$$

Now, let's redefine our axes: $J^1 \to J^2$, $J^2 \to J^3$, $J^3 \to J^1$. Alternatively, we can define: $J^1 \to J^3$, $J^2 \to J^1$, $J^3 \to J^2$. Also, $[J^i, J^i]$ obviously equals zero. Summarizing these results:

$$[J^i, J^j] = i\hbar\varepsilon^{ijk}J^k$$

This trick is called "taking cyclic permutations" and will be a standard workhorse of this course. □

Now let's consider:

$$[J^1, K^2] = \left[M^{23}, M^{20}\right]$$

Using Srednicki 2.16 again:

$$[J^1, K^2] = i\hbar\left(g^{22}M^{30} - g^{32}M^{20} + g^{30}M^{22} - g^{20}M^{32}\right)$$

The result is:

$$[J^1, K^2] = i\hbar M^{30} = i\hbar K^3$$

Using cyclic permutations:

$$[J^i, K^j] = i\hbar\varepsilon^{ijk}K^k$$

□

Finally, we consider:

$$[K^1, K^2] = \left[M^{10}, M^{20}\right]$$

$$[K^1, K^2] = i\hbar\left(g^{12}M^{00} - g^{02}M^{10} + g^{00}M^{12} - g^{10}M^{02}\right)$$

$$[K^1, K^2] = -i\hbar M^{12}$$

$$[K^1, K^2] = -i\hbar J^3$$

Taking cyclic permutations:

$$[K^i, K^j] = -i\hbar\varepsilon^{ijk}J^k$$

□

Srednicki 2.5. Verify that eq. 2.18 follows from eq. 2.15

Equation 2.15 is:

$$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$

Now let $\Lambda = 1 + \delta\omega$:

$$U(1 - \delta\omega)P^\mu U(1 + \delta\omega) = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

Using equation 2.12:

$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)P^\mu\left(1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

$$P^\mu + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}\left(P^\mu M^{\alpha\beta} - M^{\alpha\beta}P^\mu\right) = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

$$\delta\omega_{\alpha\beta}\left[P^\mu, M^{\alpha\beta}\right] = \frac{2\hbar}{i}g^{\mu\alpha}\delta\omega_{\alpha\nu}P^\nu$$

Change the dummy variable ν to β on the right hand side:

$$\delta\omega_{\alpha\beta}\left[P^\mu, M^{\alpha\beta}\right] = \frac{2\hbar}{i}g^{\mu\alpha}\delta\omega_{\alpha\beta}P^\beta$$

Now let's write right-hand side as two terms. In the second term, we'll switch $\alpha \leftrightarrow \beta$:

$$\delta\omega_{\alpha\beta}\left[P^\mu, M^{\alpha\beta}\right] = \frac{\hbar}{i}\left(g^{\mu\alpha}\delta\omega_{\alpha\beta}P^\beta + g^{\mu\beta}\delta\omega_{\beta\alpha}P^\alpha\right)$$

Now we'll remember that $\delta\omega$ is antisymmetric:

$$\delta\omega_{\alpha\beta}\left[P^\mu, M^{\alpha\beta}\right] = \frac{\hbar}{i}\left(g^{\mu\alpha}\delta\omega_{\alpha\beta}P^\beta - g^{\mu\beta}\delta\omega_{\alpha\beta}P^\alpha\right)$$

$$\delta\omega_{\alpha\beta}\left[P^\mu, M^{\alpha\beta}\right] = \frac{\hbar}{i}\delta\omega_{\alpha\beta}\left(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha\right)$$

Now we remember that $\delta\omega$ is an arbitrary anti-symmetric function (where we mean anti-symmetric under $\alpha \leftrightarrow \beta$). The anti-symmetric part of the coefficients must therefore be equal (the symmetric coefficients, on the other hand, may be different, but the equality will still hold because they go to zero when multiplied by the anti-symmetric $\delta\omega$). In this case, both coefficients are manifestly anti-symmetric, so we can equate them:

$$\left[P^\mu, M^{\alpha\beta}\right] = \frac{\hbar}{i}\left(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha\right)$$

Srednicki 2.6. Verify that eq. 2.19 follows from eq. 2.18

We'll go through these one by one. First, take:

$$[H, P_1] = [H, \frac{1}{2}M^{23}] = i\hbar c(g^{03}P^2 - g^{02}P^3) = 0$$

where the last equality follows from equation 2.18. We can replace 1 with 2 or 3, there's no way for the right hand side to be anything other than zero. □

Next we'll consider:

$$[J_1, P_2] = [\frac{1}{2}M_{23} - \frac{1}{2}M_{32}, P_2]$$

Remember that M is antisymmetric, so these terms combine. Now use 2.18:

$$[J_1, P_2] = -i\hbar(g_{23}P_2 - g_{22}P_3)$$

$$[J_1, P_2] = i\hbar P_3$$

Taking cyclic permutations:

$$[J_i, P_j] = i\hbar\varepsilon_{ijk}P_k$$

□

The third one seems simple enough that we'll proceed directly, rather than relying on cyclic permutations. We have:

$$[K_i, H] = [M_{i0}, cP_0] = -c[P_0, M_{i0}]$$

Now use 2.18:

$$[K_i, H] = -i\hbar c\left(g_{00}P_i - g_{0i}P_0\right)$$

i is a spatial variable (can't have boost in time-time direction), so:

$$[K_i, H] = i\hbar c P_i$$

□

Finally, we have:

$$[K_i, P_j] = -[P_i, M_{j0}] = -i\hbar\left(g_{i0}P_j - g_{ij}P_0\right)$$

Recalling that i is a Spatial variable, we have:

$$[K_i, P_j] = i\hbar\delta_{ij}P_0 = i\hbar\delta_{ij}H/c$$

which proves eq. 2.19. □

Srednicki 2.7. What additional constraint should be added to the translation operator, in order to prove equation 2.20?

We should require that translations are additive; that is that $T(a)T(b) = T(a+b)$. Hence, it doesnt matter which order translations are performed in (including translations in time), the result will be the same.

To see that this will allow us to prove equation 2.20, recall that momentum is the generator of translation:

$$f(x+a) = T(a)f(x) = exp\left(\frac{iap}{\hbar}\right)f(x) = f(x) + \left(\frac{iap}{\hbar}\right)f(x) + \ldots$$

This was proved in the chapter summary; see also Griffiths' Quantum book, problem 3.39. From this, we see that the infinitesimal translation operator is simply $iap/\hbar$.

Now we'll imagine two infinitesimal translations in two different directions μ and ν. Obviously

$$T(\delta_1 + \delta_2) = T(\delta_2 + \delta_1)$$

Using the additive property described previously:

$$T(\delta_1)T(\delta_2) = T(\delta_2)T(\delta_1)$$

Now use the translation operator for infinitesimals:

$$(iap_m u/\hbar)(ibp_n u/\hbar) = (iap_n u/\hbar)(ibp_m u/\hbar)$$

Canceling the constants, we have

$$p_\mu p_\nu = p_\nu p_\mu$$

If these are both space directions, then $[p_i, p_j] = 0$. If one of them is a time direction, then $[H, p_i] = 0$. This proves equation 2.20.

Srednicki 2.8. (a) Let $\mathbf{\Lambda = 1 + \delta\omega}$ in eq. 2.26 and show that

$$[\phi(\mathbf{x}), \mathbf{M}^{\mu\nu}] = \mathcal{L}^{\mu\nu}\phi(\mathbf{x})$$

where

$$\mathcal{L}^{\mu\nu} = \frac{\hbar}{\mathbf{i}}(\mathbf{x}^\mu\partial^\nu - \mathbf{x}^\nu\partial^\mu)$$

Equation 2.26 gives:

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

Taking $\Lambda = 1 + \delta\omega$ as indicated gives:

$$(1 - \delta\omega)\phi(x)(1 + \delta\omega) = \phi(x - x\delta\omega)$$

On the left, we'll use equation 2.12; on the right, we'll expand:

$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\phi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) = \phi(x) - \delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)$$

Multiplying these out, we get:

$$\phi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}\left(\phi(x)M^{\mu\nu} - M^{\mu\nu}\phi(x)\right) = \phi(x) - \delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)$$

$$\frac{i}{2\hbar}\delta\omega_{\mu\nu}\left(\phi(x)M^{\mu\nu} - M^{\mu\nu}\phi(x)\right) = -\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)$$

which gives:

$$\frac{i}{2\hbar}\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = -\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)$$

We're getting somewhere; the problem is that the coefficient of $\delta\omega$ on the right side is not symmetric or antisymmetric, so we will have difficulty killing the $\delta\omega$ term. Let's rewrite the right hand side as two terms, where in the second we'll swap the dummy indices:

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = -\frac{\hbar}{i}\left(\delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x) + \delta\omega_{\nu\mu}x^{\mu}\partial^{\nu}\phi(x)\right)$$

Now in the second term, we'll remember that $\delta\omega$ is antisymmetric, so that we can factor it out. We'll also multiply through the minus sign and factor out a $\phi(x)$:

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = \frac{\hbar}{i}\delta\omega_{\mu\nu}\left(-x^{\nu}\partial^{\mu} + x^{\mu}\partial^{\nu}\right)\phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = \delta\omega_{\mu\nu}\mathcal{L}\phi(x)$$

Now, $\delta\omega$ is an arbitrary anti-symmetric function, so the antisymmetric part of the coefficients on both sides must be equal. In this case both sides have coefficients which are manifestly anti-symmetric (antisymmetric in the same way that $\delta\omega$ is, in this case, under $\mu \leftrightarrow \nu$), so:

$$[\phi(x), M^{\mu\nu}] = \mathcal{L}\phi(x)$$

as expected.

(b) Show that $[[\phi(\mathbf{x}), \mathbf{M}^{\mu\nu}], \mathbf{M}^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(\mathbf{x})$.

The key insight we need is that $\mathcal{L}$ and M commute with one another. This is because $\mathcal{L}$ is an operator acting on x, while M is an operator acting in Hilbert Space, and it cannot be written as a function of x. With this fact, we can proceed directly. Using the result from part (a):

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\phi(x), M^{\rho\sigma}]$$

Using the properties of the commutator:

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}[\phi(x), M^{\rho\sigma}] + \phi(x)[\mathcal{L}^{\mu\nu}, M^{\rho\sigma}]$$

As discussed above, the second term vanishes. For the first, we again use our result from part (a):

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x)$$

as expected.

(c) Prove the Jacobi Identity: $[[A,B],C]+[[B,C],A]+[[C,A],B]=0$ **.**

This is a thrilling and insightful proof. We rewrite the left hand side as:

$$[AB,C]-[BA,C]+[BC,A]-[CB,A]+[CA,B]-[AC,B]$$

Now we use the property of the commutator to find:

$$A[B,C]+[A,C]B-B[A,C]-[B,C]A+B[C,A]+[B,A]C$$

$$-C[B,A]-[C,A]B+C[A,B]+[C,B]A-A[C,B]-[A,B]C$$

None of these cancel, so we must expand further:

$$ABC-ACB+ACB-CAB-BAC+BCA-BCA+CBA+BCA-BAC+BAC-ABC$$

$$-CBA+CAB-CAB+ACB+CAB-CBA+CBA-BCA-ACB+ABC-ABC+BAC$$

These cancel in pairs, leaving zero.

(d) Use your results from parts (b) and (c) to show that

$$[\phi(\mathbf{x}),[\mathbf{M}^{\mu\nu},\mathbf{M}^{\rho\sigma}]]=\left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(\mathbf{x})$$

We have, rewriting slightly:

$$[\phi(x),[M^{\mu\nu},M^{\rho\sigma}]]=-[[M^{\mu\nu},M^{\rho\sigma}],\phi(x)]$$

Now use the Jacobi Identity (part c):

$$[\phi(x),[M^{\mu\nu},M^{\rho\sigma}]]=[[M^{\mu\nu},\phi(x)],M^{\rho\sigma}]+[[\phi(x),M^{\mu\nu}],M^{\rho\sigma}]$$

which becomes:

$$[\phi(x),[M^{\mu\nu},M^{\rho\sigma}]]=-[[\phi(x),M^{\mu\nu}],M^{\rho\sigma}]+[[\phi(x),M^{\mu\nu}],M^{\rho\sigma}]$$

Now we use the result from part (b), factor out a ϕ and we have:

$$[\phi(x),[M^{\mu\nu},M^{\rho\sigma}]]=\left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(x)$$

as expected.

(e) Simplify the right hand of equation 2.31 as much as possible.

Let's start by recalling how index notation works. Consider:

$$(x^\mu\partial^\nu)(x^\rho\partial^\sigma)=x^\mu x^\rho\partial^\nu\partial^\sigma+x^\mu\partial^\sigma\partial^\nu x^\rho$$

Now, what to do with the unevaluated derivative? Notice:

$$\partial^\nu x^\rho=\begin{cases}0 & \text{if} \quad \nu\neq\rho\\ -1 & \text{if} \quad \nu=\rho=0\\ 1 & \text{if} \quad \nu=\rho=1\end{cases}=g^{\nu\rho} \tag{2.8.1}$$

because $\partial^0 x^0 = -\frac{1}{c}\frac{\partial}{\partial t}ct = -1$, whereas $\partial^i x^i = \frac{\partial}{\partial x^i}x^i = 1$. Hence, our result is:

$$(x^\mu\partial^\nu)(x^\rho\partial^\sigma) = x^\mu x^\rho \partial^\nu\partial^\sigma + x^\mu\partial^\sigma g^{\nu\rho} \tag{2.8.2}$$

We proceed. The right-hand side of eq. 2.31 is:

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = -\hbar^2\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)(x^\rho\partial^\sigma - x^\sigma\partial^\rho) - (x^\rho\partial^\sigma - x^\sigma\partial^\rho)(x^\mu\partial^\nu - x^\nu\partial^\mu)\right]\phi(x)$$

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = -\hbar^2\left[x^\mu\partial^\nu(x^\rho\partial^\sigma) - x^\mu\partial^\nu(x^\sigma\partial^\rho) - x^\nu\partial^\mu(x^\rho\partial^\sigma) + x^\nu\partial^\mu(x^\sigma\partial^\rho)\right.$$
$$\left. - x^\rho\partial^\sigma(x^\mu\partial^\nu) + x^\rho\partial^\sigma(x^\nu\partial^\mu) + x^\sigma\partial^\rho(x^\mu\partial^\nu) - x^\sigma\partial^\rho(x^\nu\partial^\mu)\right]\phi(x)$$

Now we use equation (2.8.2) eight times:

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = -\hbar^2\left[x^\mu x^\rho\partial^\nu\partial^\sigma + x^\mu\partial^\sigma g^{\nu\rho} - x^\mu x^\sigma\partial^\nu\partial^\rho - x^\mu\partial^\rho g^{\nu\sigma} - x^\nu x^\rho\partial^\mu\partial^\sigma\right.$$
$$-x^\nu\partial^\sigma g^{\mu\rho} + x^\nu x^\sigma\partial^\mu\partial^\rho + x^\nu\partial^\rho g^{\mu\sigma} - x^\rho x^\mu\partial^\sigma\partial^\nu - x^\rho\partial^\nu g^{\sigma\mu} + x^\rho x^\nu\partial^\sigma\partial^\mu$$
$$\left.+x^\rho\partial^\mu g^{\sigma\nu} + x^\sigma x^\mu\partial^\rho\partial^\nu + x^\sigma\partial^\nu g^{\rho\mu} - x^\sigma x^\nu\partial^\rho\partial^\mu - x^\sigma\partial^\mu g^{\rho\nu}\right]\phi(x)$$

Fortunately, the terms without g's cancel, leaving:

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = -\hbar^2\left[x^\mu\partial^\sigma g^{\nu\rho} - x^\mu\partial^\rho g^{\nu\sigma} - x^\nu\partial^\sigma g^{\mu\rho} + x^\nu\partial^\rho g^{\mu\sigma} - x^\rho\partial^\nu g^{\sigma\mu}\right.$$
$$\left.+x^\rho\partial^\mu g^{\sigma\nu} + x^\sigma\partial^\nu g^{\rho\mu} - x^\sigma\partial^\mu g^{\nu\rho}\right]\phi(x)$$

Grouping and factoring:

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = -\hbar^2\left[g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - g^{\nu\sigma}(x^\mu\partial^\rho - x^\rho\partial^\mu) - g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu)\right.$$
$$\left.+g^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu)\right]\phi(x)$$

which is, after using the definition of $\mathcal{L}$ and reordering a bit:

$$[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x) = i\hbar\left[g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho} + g^{\nu\sigma}\mathcal{L}^{\mu\rho}\right]\phi(x) \tag{2.8.3}$$

which looks exactly like equation 2.16, only for the $\mathcal{L}$'s.

(f) Use your results from part (e) to verify equation 2.16, up to the possibility of a term on the right hand side that commutes with ϕ(x) and its derivatives. (Such a term, called a *central charge*, in fact does not arise for the Lorentz Algebra.)

Start with equation 2.31:

$$[\phi(x),[M^{\mu\nu},M^{\rho\sigma}]] = [\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}]\phi(x)$$

On the right, we'll use equation (2.8.3); on the left, we'll assume the form of 2.16, but we'll also add an arbitrary k so that we don't assume the thing we're trying to prove. Then:

$$i\hbar[\phi(x), g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} + k] = [g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho} + g^{\nu\sigma}\mathcal{L}^{\mu\rho}]\phi(x)$$

which gives:

$$i\hbar\left(g^{\mu\rho}[\phi(x),M^{\nu\sigma}] - g^{\nu\rho}[\phi(x),M^{\mu\sigma}] + g^{\nu\sigma}[\phi(x),M^{\mu\rho}] - g^{\mu\sigma}[\phi(x),M^{\nu\rho}] + [\phi(x),k]\right)$$

$$= [g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho} + g^{\nu\sigma}\mathcal{L}^{\mu\rho}]\phi(x)$$

Now we simplify using equation 2.29:

$$i\hbar\left(g^{\mu\rho}\mathcal{L}^{\nu\sigma}\phi(x) - g^{\nu\rho}\mathcal{L}^{\mu\sigma}\phi(x) + g^{\nu\sigma}\mathcal{L}^{\mu\rho}\phi(x) - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\phi(x) + [\phi(x), k]\right)$$
$$= [g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho} + g^{\nu\sigma}\mathcal{L}^{\mu\rho}]\phi(x)$$

Most terms cancel, we're left with:

$$[\phi(x), k] = 0$$

Hence, k is no longer an arbitrary function, but an arbitrary function that commutes with ϕ and its derivatives. This of course is the central charge that was referred to previously, and works out to be zero. By the way, we already know $k = 0$ due to problem 2.3, but that proof relies upon the assumption that $U(\Lambda^{-1}) = U(\Lambda)^{-1}$. Other sources will prove that the central charge is zero without any troubling assumptions.

Srednicki 2.9. Let us write:

$$\mathbf{\Lambda}^{\rho}{}_{\tau} = \delta^{\rho}{}_{\tau} + \frac{\mathbf{i}}{\mathbf{2\hbar}}\delta\omega_{\mu\nu}(\mathbf{S}^{\mu\nu}_{\mathbf{V}})^{\rho}{}_{\tau}$$

where

$$(\mathbf{S}^{\mu\nu}_{\mathbf{V}})^{\rho}{}_{\tau} = \frac{\mathbf{\hbar}}{\mathbf{i}}(\mathbf{g}^{\mu\rho}\delta^{\nu}{}_{\tau} - \mathbf{g}^{\nu\rho}\delta^{\mu}{}_{\tau})$$

are matrices which constitute the *vector representation* of the Lorentz generators.

(a) Let $\mathbf{\Lambda} = \mathbf{1} + \delta\omega$ in eq. 2.27 and show that

$$[\partial^{\mu}\phi(\mathbf{x}), \mathbf{M}^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^{\rho}\phi(\mathbf{x}) + (\mathbf{S}^{\mu\nu}_{\mathbf{V}})^{\rho}{}_{\tau}\partial^{\tau}\phi(\mathbf{x})$$

Let's go to equation 2.27 as indicated:

$$U(\Lambda)^{-1}\partial^{\mu}\phi(x)U(\Lambda) = \partial^{\mu}\phi(\Lambda^{-1}x)$$

Let's start by pulling a derivative out of the left hand side. This gives:

$$\partial^{\mu}\left[U(\Lambda)^{-1}\phi(x)U(\Lambda)\right] = \partial^{\mu}\left[\phi(\Lambda^{-1}x)\right]$$

Now we're ready to let $\Lambda = 1 + \delta\omega$. But we've already done this! Except for the presence of the derivative, this is exactly the supposition we made at the outset of problem 2.8a; hence we'll just quote the result (eq. 2.29), and insert the derivatives (if you didn't do problem 2.8a, there's nothing for it but to solve the problem now, adding a derivative to both sides of each equation).

$$\partial^{\mu}[\phi(x), M^{\alpha\beta}] = \partial^{\mu}\mathcal{L}^{\alpha\beta}\phi(x)$$

We recall that M is an operator in Hilbert space and does not depend on x, so we may assume that it commutes with ∂^{μ}. Hence,

$$[\partial^{\mu}\phi(x), M^{\alpha\beta}] = \partial^{\mu}\mathcal{L}^{\alpha\beta}\phi(x)$$

Now there are a few ways we could proceed, but I think it's easiest to avoid logical fallicies if we proceed by using the product rule. Then,

$$[\partial^{\mu}\phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^{\mu}\phi(x) + \left(\partial^{\mu}\mathcal{L}^{\alpha\beta}\right)\phi(x)$$

The first term looks good, the second term will be simplified. But we need to remember this rule of basic calculus: *the* ∂^μ *acts only on the* $\mathcal{L}$*. If the derivative is able to commute all the way through the* $\mathcal{L}$*, it will act on the constant and give zero. It will never act on the* $\phi(x)$. This is just the product rule, but it's easy to mix it up, especially if you try to deal with commutators in the abstract.

In any case, we write out $\mathcal{L}$:

$$[\partial^\mu \phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^\mu\phi(x) + \frac{\hbar}{i}\left(\partial^\mu(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\right)\phi(x)$$

$$[\partial^\mu \phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^\mu\phi(x) + \frac{\hbar}{i}\left(\partial^\mu(x^\alpha)\partial^\beta - \partial^\mu(x^\beta)\partial^\alpha\right)\phi(x)$$

Now we take the derivatives (using (2.8.1)) and we have:

$$[\partial^\mu \phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^\mu\phi(x) + \frac{\hbar}{i}\left(g^{\mu\alpha}\partial^\beta - g^{\mu\beta}\partial^\alpha\right)\phi(x)$$

Now we'll introduce a new index, let's call it τ:

$$[\partial^\mu \phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^\mu\phi(x) + \frac{\hbar}{i}\left(g^{\alpha\mu}\delta^\beta_{\ \tau} - g^{\beta\mu}\delta^\alpha_{\ \tau}\right)\partial^\tau\phi(x)$$

which is:

$$[\partial^\mu \phi(x), M^{\alpha\beta}] = \mathcal{L}^{\alpha\beta}\partial^\mu\phi(x) + (S_V^{\alpha\beta})^\mu_{\ \tau}\partial^\tau\phi(x)$$

as expected.

(b) Show that the matrices $\mathbf{S_V^{\mu\nu}}$ have the same commutation relations as the operators $\mathbf{M^{\mu\nu}}$. Hint: see the previous problem.

Srednicki's hint is non-rigorous. In the last step, one has to assume that the commutator of S_V involves no terms that involve $\mathcal{L}$ (and vice-versa). This is perhaps a reasonable assumption, but the straightforward approach, though tedious, is rigorous.

So, the commutator of S_V is:

$$[S_V^{\mu\nu}, S_V^{\alpha\beta}] = \frac{\hbar^2}{i^2}\left[(S_V^{\mu\nu})^\gamma_{\ \sigma}(S_V^{\alpha\beta})^{\sigma\delta} - (S_V^{\mu\nu})^\delta_{\ \sigma}(S_V^{\alpha\beta})^{\sigma\gamma}\right]$$

Using the definition of S_V (I'll suppress the left-hand side since nothing happens to it):

$$\frac{\hbar^2}{i^2}\left[(g^{\mu\gamma}\delta^\nu_{\ \sigma} - g^{\nu\gamma}\delta^\mu_{\ \sigma})(g^{\alpha\sigma}\delta^{\beta\delta} - g^{\beta\sigma}\delta^{\alpha\delta}) - (g^{\mu\delta}\delta^\nu_{\ \sigma} - g^{\nu\delta}\delta^\mu_{\ \sigma})(g^{\alpha\sigma}\delta^{\beta\gamma} - g^{\beta\sigma}\delta^{\alpha\gamma})\right]$$

Expanding, using one δ, reordering, and grouping, gives:

$$\frac{\hbar^2}{i^2}\left[g^{\alpha\nu}\left(g^{\gamma\mu}\delta^{\delta\beta} - g^{\delta\mu}\delta^{\gamma\beta}\right) + g^{\beta\nu}\left(g^{\delta\mu}\delta^{\gamma\alpha} - g^{\gamma\mu}\delta^{\delta\alpha}\right) + g^{\alpha\mu}\left(g^{\delta\nu}\delta^{\gamma\beta} - g^{\gamma\nu}\delta^{\delta\beta}\right) + g^{\mu\beta}\left(g^{\gamma\nu}\delta^{\delta\alpha} - g^{\delta\nu}\delta^{\gamma\alpha}\right)\right]$$

which is:

$$\frac{\hbar}{i}\left[g^{\alpha\nu}(S_V^{\gamma\delta})^{\mu\beta} + g^{\beta\nu}(S_V^{\delta\gamma})^{\mu\alpha} + g^{\alpha\mu}(S_V^{\delta\gamma})^{\nu\beta} + g^{\mu\beta}(S_V^{\gamma\delta})^{\nu\alpha}\right]$$

Remember (or notice) that S_V is anti-symmetric. Hence:

$$i\hbar\left[-g^{\alpha\nu}(S_V^{\gamma\delta})^{\mu\beta}+g^{\beta\nu}(S_V^{\gamma\delta})^{\mu\alpha}+g^{\alpha\mu}(S_V^{\gamma\delta})^{\nu\beta}-g^{\mu\beta}(S_V^{\gamma\delta})^{\nu\alpha}\right]$$

Now, the γ and δ markers aren't very illuminating, so let's suppress them. We'll also reorder and reinsert the left-hand side:

$$[S_V^{\mu\nu},S_V^{\alpha\beta}]=i\hbar\left[g^{\alpha\mu}S_V^{\nu\beta}-g^{\alpha\nu}S_V^{\mu\beta}-g^{\mu\beta}S_V^{\nu\alpha}+g^{\beta\nu}S_V^{\mu\alpha}\right]$$

which is the same relation as 2.16.

(c) For a rotation by an angle θ about the z axis, we have:

$$\Lambda^{\mu}_{\ \nu}=\begin{pmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{cos}\ \theta & -\mathbf{sin}\ \theta & \mathbf{0}\\ \mathbf{0} & \mathbf{sin}\ \theta & \mathbf{cos}\ \theta & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix}$$

Show that

$$\boldsymbol{\Lambda}=\mathbf{exp}(-\mathbf{i}\theta\mathbf{S}_{\mathbf{V}}^{\mathbf{12}}/\mathbf{\hbar})$$

First we'll work out S_V^{12}:

$$(S_V^{12})^{\rho}_{\ \tau}=\frac{\hbar}{i}(g^{1\rho}\delta^{2}_{\ \tau}-g^{2\rho}\delta^{1}_{\ \tau})$$

This works out to be:

$$S_V^{12}=\frac{\hbar}{i}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}$$

For notational convenience, we'll define

$$A=\frac{i}{\hbar}S_V^{12}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}$$

Now $exp(-i\theta S_V^{12}/\hbar)=e^{-\theta A}$, so we can expand in a Taylor Series:

$$e^{-\theta A}=1-\theta A+\frac{1}{2!}\theta^2A^2-\frac{1}{3!}\theta^3A^3+\frac{1}{4!}\theta^4A^4-\frac{1}{5!}\theta^5A^5+\frac{1}{6!}\theta^6A^6+\ldots$$

Now, note that

$$A^2=-\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

Hence, $AA^2=-A$, so we can simplify our expansion:

$$e^{-\theta A}=1-\theta A+\frac{1}{2!}\theta^2A^2+\frac{1}{3!}\theta^3A-\frac{1}{4!}\theta^4A^2-\frac{1}{5!}\theta^5A+\frac{1}{6!}\theta^6A^2+\ldots$$

$$e^{-\theta A} = 1 + \left(-\theta + \frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \ldots\right) A + \left(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \frac{1}{6!}\theta^6 + \ldots\right) A^2$$

$$e^{-\theta A} = 1 - (sin\ \theta)A + (1 - cos\ \theta)A^2$$

From which we plug in A and see:

$$e^{-i\theta S_V^{12}/\hbar} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\ \theta & -sin\ \theta & 0 \\ 0 & sin\ \theta & cos\ \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda$$

as expected.

(d) For a boost by *rapidity* η in the z direction, we have:

$$\mathbf{\Lambda}^{\mu}{}_{\nu} = \begin{pmatrix} \mathbf{cosh}\ \eta & \mathbf{0} & \mathbf{0} & \mathbf{sinh}\ \eta \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{sinh}\eta & \mathbf{0} & 0 & \mathbf{cosh}\ \eta \end{pmatrix}$$

Show that

$$\mathbf{\Lambda = exp(i\eta S_V^{30}/\hbar)}$$

First we'll work out S_V^{30}:

$$(S_V^{30})^{\rho}{}_{\tau} = \frac{\hbar}{i}(g^{3\rho}\delta^0{}_{\tau} - g^{0\rho}\delta^3{}_{\tau})$$

This works out to be:

$$S_V^{30} = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For notational convenience, we'll define

$$A = \frac{i}{\hbar}S_V^{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now $exp(i\eta S_V^{30}/\hbar) = e^{\eta A}$, so we can expand in a Taylor Series:

$$e^{\eta A} = 1 + \eta A + \frac{1}{2!}\eta^2 A^2 + \frac{1}{3!}\eta^3 A^3 + \frac{1}{4!}\eta^4 A^4 + \frac{1}{5!}\eta^5 A^5 + \ldots$$

Now, note that

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Further, $AA^2 = A$, so we can simplify our expansion:

$$e^{\eta A} = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A + \frac{1}{4!}A^2 + \frac{1}{5!}A + \ldots$$

$$e^{\eta A} = 1 + \left(\eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \dots\right) A + \left(\frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \dots\right) A^2$$

$$e^{\eta A} = 1 + (\sinh\ \eta)A + (\cosh\ \eta - 1)A^2$$

From which we plug in A and see:

$$exp(i\eta S_V^{30}/\hbar) = \begin{pmatrix} cosh\ \eta & 0 & 0 & sinh\ \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sinh\ \eta & 0 & 0 & cosh\ \eta \end{pmatrix} = \Lambda$$

as expected.

Problem 2.10. Prove that the Lorentz Group is a group.

Closure. Let's assume that Λ' and Λ'' are Lorentz Matrices. We must prove that their product is also in the group. In index notation, the product is: $\Lambda^\mu_\alpha = \Lambda'^\mu_\gamma \Lambda''^\gamma_\alpha$. Now, we want to consider the following:

$$g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\mu\nu}\Lambda'^\mu_\gamma \Lambda''^\gamma_\alpha \Lambda'^\nu_\delta \Lambda''^\delta_\beta$$

These are in index notation, so we can move stuff around without worrying about commutation relations.

$$g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\mu\nu}\Lambda'^\mu_\gamma \Lambda'^\nu_\delta \Lambda''^\gamma_\alpha \Lambda''^\delta_\beta$$

Now Λ' is in the group, so by the definition of the group:

$$g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\gamma\delta}\Lambda''^\gamma_\alpha \Lambda''^\delta_\beta$$

Λ'' is also in the group, so we do this trick again:

$$g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}$$

and so Λ is a member of the group, and therefore the group is closed. □

Inversion. We can rewrite the definition of the group:

$$\Lambda_{\nu\rho}\Lambda^\nu_\sigma = g_{\rho\sigma}$$

Raise the ρ index on both sides:

$$\Lambda^\rho_\nu \Lambda^\nu_\sigma = g^\rho_\sigma$$

which is

$$\Lambda^\rho_\nu \Lambda^\nu_\sigma = \delta^\rho_\sigma$$

On the other hand, by definition,

$$(\Lambda^{-1})^\rho{}_\nu \Lambda^\nu{}_\sigma = \delta^\rho{}_\sigma$$

From which it follows that:

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda^\rho_\nu$$

and so the inverse exists and is a member of the group. □

Identity. The 4x4 identity matrix will do, we just have to show that it's a member of the group.

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\mu\nu}\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

which is the definition of the group. Thus, the identity exists. $\square$

Association. The group's operator is matrix multiplication, which is associative. $\square$

Problem 2.11. Show that propriety and orthochronaity are subgroups of the Lorentz Group.

It should be obvious that these are subsets of the Lorentz Group; it remains to show that they are groups in their own right.

We've already proved that all Lorentz Group Members have inverses and an associative operator. The proposed subgroups use the same operator as the groups themselves, so associativity is fine.

Rewriting the inverse equation (Srednicki 2.5), we have: $(\Lambda^{\alpha}{}_{\beta})^{-1} = \Lambda_{\beta}{}^{\alpha} = \Lambda^{\sigma}{}_{\gamma}g_{\sigma\beta}g^{\gamma\alpha}$. Taking the determinant of both sides shows that the inverse has the same determinant as the original function. Taking the zero-zero component shows that $\Lambda^{0}{}_{0}$ will be the same for the original Λ as well as its inverse. Hence, inverses are members of the subgroup.

The identity is the 4x4 identity matrix, which is proper and orthochronous. So the identity is in the subgroup.

It remains to show closure. The product of two matrices with determinant 1 will have determinant 1, and so propriety is closed.

The hard part of this problem is proving that orthochronality is closed. First, we note that inverses will be included in the subgroup, so for any orthochronous Lorentz Matrix:

$$g_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

Now we use the definition of the inverse:

$$g_{\mu\nu}\Lambda_{\alpha}{}^{\mu}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

Playing with the indices, we can put this in the form:

$$\Lambda^{\alpha}{}_{\mu}\Lambda^{\mu\beta} = g^{\alpha\beta}$$

From which it follows that:

$$\Lambda^{0}{}_{\mu}\Lambda^{\mu 0} = g_{\mu\nu}\Lambda^{0\nu}\Lambda^{\mu 0} = g^{00} = -1$$

Further:

$$-\Lambda^{00}\Lambda^{00} + \sum_{i=0}^{3}\Lambda^{0i}\Lambda^{i0} = -1$$

hence:

$$\sum_{i=1}^{3} \Lambda^{0i}\Lambda^{i0} = -1 + \Lambda^{00}\Lambda^{00} \leq \Lambda^{00}\Lambda^{00} \tag{2.11.1}$$

Now let's imagine the product of two orthochronous Lorentz Matrices:

$$\Lambda = AB$$

$$\Lambda^{\mu}_{\ \nu} = g_{\alpha\nu} A^{\mu\beta} B_{\beta}^{\ \alpha}$$

Hence:

$$\Lambda^{0}_{\ 0} = g_{00} A^{0\beta} B_{\beta}^{\ 0}$$
$$\Lambda^{0}_{\ 0} = -A^{0\beta} B_{\beta}^{\ 0}$$
$$\Lambda^{0}_{\ 0} = -g_{\delta\beta} A^{0\beta} B^{\delta 0}$$
$$\Lambda^{0}_{\ 0} = A^{00}B^{00} - \sum_{i=1}^{3} A^{0i}B^{i0}$$

By the Schwartz Inequality (with the inner product of A and B defined such that its square magnitude is given on the left side of the equation),

$$|\sum_{i=1}^{3} A^{0i}B^{i0}|^2 \leq \sum_{i=1}^{3} |A^{0i}A^{i0}| \sum_{j=1}^{3} |B^{0j}B^{j0}|$$

Using equation (2.11.1):

$$|\sum_{i=1}^{3} A^{0i}B^{i0}|^2 \leq (A^{00})^2(B^{00})^2$$

Thus, in our definition for Λ^0_0, the second term has less magnitude than the first term. Hence, it follows that the second term cannot reverse the sign of the first term. Nor can it make the sign zero, since we proved already that $|\Lambda^0_0| \geq 1$. Hence, the sign is completely determined by the first term, from which it is obvious that Λ will be orthochronous if A and B are.

As a fun fact, notice that if A and B are both non-orthochronous, then their product will be orthochronous. This means that non-orthochronality is not closed, and is therefore not a group (it's also not a group because it has no identity). Conversely, if A is orthochronous and B is not, then the product will be non-orthochronous (which proves nothing about the group, but might be useful elsewhere).