Srednicki Chapter 16

QFT Problems & Solutions

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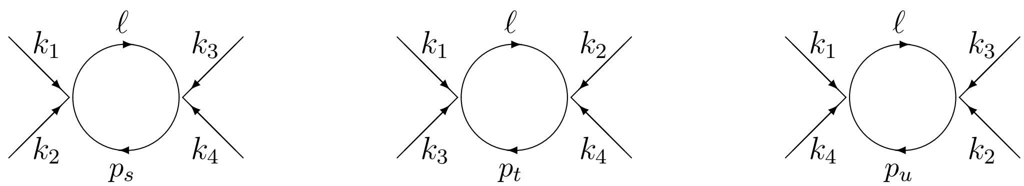
Srednicki 16.1. Compute the O($\boldsymbol{\lambda^2}$) correction to $\boldsymbol{V_4}$ in $\boldsymbol{\phi^4}$ theory (see problem 9.2). Take $\mathbf{V_4 = \boldsymbol{\lambda}}$ when all four external momenta are on shell, and $\mathbf{s = 4m^2}$. What is the O($\boldsymbol{\lambda}$) contribution to $\boldsymbol{C}$?

In ϕ^4 theory, the first-order correction to the vertex is:



where, by conservation of momentum, $p_s = \ell - k_1 - k_2$, $p_t = \ell - k_1 - k_3$, $p_u = \ell - k_1 - k_4$.

Assessing the values of these diagrams in the usual way, we have:

$$iV_4 = -iZ_\lambda\lambda + \frac{1}{2}\left(-iZ_\lambda\lambda\right)^2\left(\frac{1}{i}\right)^2\int\frac{d^d\ell}{(2\pi)^d}\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_2)^2)$$

$$+\frac{1}{2}\left(-iZ_\lambda\lambda\right)^2\left(\frac{1}{i}\right)^2\int\frac{d^d\ell}{(2\pi)^d}\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_3)^2)$$

$$+\frac{1}{2}\left(-iZ_\lambda\lambda\right)^2\left(\frac{1}{i}\right)^2\int\frac{d^d\ell}{(2\pi)^d}\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_4)^2) + O(\lambda^4) \qquad (16.1.1)$$

Now let's consider the two propagators in the second term, using Feynman's Formula:

$$\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_2)^2) = \int dF_2\left[x_1\ell^2 + x_2(\ell-k_1-k_2)^2 + m^2\right]^{-2}$$

which is:

$$\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_2)^2) = \int_0^1 dx_1dx_2\left[x_1\ell^2 + x_2(\ell-k_1-k_2)^2 + m^2\right]^{-2}\delta(x_1+x_2-1)$$

Now we do the x_1 integral, and rename $x_2 \to x$. We find:

$$\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell-k_1-k_2)^2) = \int_0^1 dx\left[\ell^2 - x\ell^2 + x\ell^2 + xk_1^2 + xk_2^2 - 2\ell xk_1 - 2\ell k_2x + 2k_1k_2x + m^2\right]^{-2}$$

Now we define $q = \ell - k_1 x - k_2 x$. Then,

$$\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell - k_1 - k_2)^2) = \int_0^1 dx \left[q^2 - x^2(k_1^2 + 2k_1 k_2 + k_2^2) + xk_2^2 + xk_1^2 + 2k_1 k_2 x + m^2\right]^{-2}$$

Now we define $D = x(1-x)(k_1 + k_2)^2 + m^2$. Then:

$$\tilde{\Delta}(\ell^2)\tilde{\Delta}((\ell - k_1 - k_2)^2) = \int_0^1 dx \left[q^2 + D\right]^{-2}$$

Now let's write out the full second term of equation (16.1.1):

$$iV_{4,\text{ term2}} = \frac{1}{2}\lambda^2 \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx (q^2 + D)^{-2}$$

Now we make the Wick Rotation:

$$iV_{4,\text{ term2}} = \frac{i}{2}\lambda^2 \int \frac{d^d \bar{q}}{(2\pi)^d} \int_0^1 dx (\bar{q}^2 + D)^{-2}$$

We can see that this integral divergs for $d \geq 4$. Evaluating the integral for $d < 4$, the result from equation 14.27 is:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx\ \Gamma\left(2 - \frac{d}{2}\right) D^{-2+d/2}$$

Now we define $\varepsilon = 4 - d$, so that we can take the limit as $d \to 4$ (equivalently, as $\varepsilon \to 0$):

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^{2-\varepsilon/2}} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx\ D^{-\varepsilon/2}$$

Now we move the mass dimensionality onto $\tilde{\mu}$ (ie define $\lambda \to \lambda\tilde{\mu}^{\varepsilon/2}$):

$$iV_{4,\text{ term2}} = \frac{i\lambda^2 \tilde{\mu}^{\varepsilon}}{2(4\pi)^{2-\varepsilon/2}} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx\ D^{-\varepsilon/2}$$

which gives:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx\ \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon/2}$$

Now we take $\varepsilon \to 0$:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2} \left(\frac{2}{\varepsilon} - \gamma\right) \int_0^1 dx\ \left[1 + \frac{\varepsilon}{2} ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)\right]$$

Expanding:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2} \int_0^1 dx\ \left[\frac{2}{\varepsilon} - \gamma + ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)\right]$$

Now we define: $4\pi\tilde{\mu}^2 = e^{\gamma}\mu^2$. Then:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2} \int_0^1 dx\ \left[\frac{2}{\varepsilon} - \gamma + ln\left(\frac{e^{\gamma}\mu^2}{D}\right)\right]$$

Breaking up the logarithm, we have:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx\ \left[\frac{2}{\varepsilon} + ln\left(\frac{\mu^2}{D}\right)\right]$$

Using the explicit form of D, and integrating the first term:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + \int_0^1 dx\ ln\left(\frac{\mu^2}{x(1-x)(k_1+k_2)^2+m^2}\right)\right] \tag{16.1.2}$$

Note that we have $(k_1+k_2)^2 = -s = -4m^2$, where this last equality is due to our normalization condition. So:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + \int_0^1\ dx\ ln\left(\frac{\mu^2}{-x(1-x)m^2+m^2}\right)\right]$$

which is:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + \int_0^1 dx\ ln\left(\frac{\mu^2}{(1-2x)^2m^2}\right)\right]$$

Breaking up the logarithm:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 2\int_0^1 dx\ ln\left(\frac{\mu/m}{1-2x}\right)\right]$$

This integral can be done by hand, the result is:

$$iV_{4,\text{ term2}} = \frac{i\lambda^2}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + 1\right] \tag{16.1.3}$$

What about the third and fourth terms of equation (16.1.1)? Going back to equation (16.1.2), we take the substitution $k_2 \to k_3$ (for t-channel) and $k_2 \to k_4$ (for u-channel). Our on-shell condition requires that $t = u = 0$. Then, both the third and fourth terms are given by:

$$iV_{4,\text{ terms 3 \& 4}} = \frac{i\lambda^2}{(4\pi)^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right)\right] \tag{16.1.4}$$

where the integral is trivial since the x terms vanish.

Now we go back to the equation (16.1.1):

$$iV_4 = -iZ_\lambda\lambda + \frac{i\lambda^2}{16\pi^2}\left[\frac{3}{\varepsilon} + 3ln\left(\frac{\mu}{m}\right) + 1\right] + O(\lambda^4)$$

Hence,

$$V_4 = -Z_\lambda\lambda + \frac{3\lambda^2}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] + O(\lambda^4)$$

We define $Z_\lambda = 1 + C$, where C is at least of order λ (we know it can't be of lower order, since any terms of lower order will be set to zero by the normalization of V_4). Then,

$$V_4 = -\lambda - C\lambda + \frac{3\lambda^2}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] + O(\lambda^4)$$

Now, we don't like how this is dependent on μ, and infinite! To avoid this, we subsume these terms into C. Hence, we take:

$$C = \frac{3\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] - \kappa_C + O(\lambda^2)$$

This gives:

$$V_4 = -\lambda + \kappa_C + O(\lambda^4)$$

Using our analogy to equation 16.14, we take $\kappa_C = 0$. Then,

$$V_4 = -\lambda + O(\lambda^4)$$

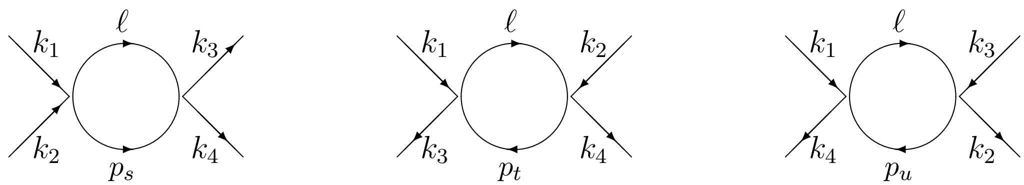
and:

$$C = \frac{3\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] + O(\lambda^2)$$

Note: There is a typo in the problem statement, see Srednicki's errata.

Srednicki 16.2. Repeat problem 16.1 for the theory of problem 9.3.

It is helpful to refer back to the solution to problem 10.2. The diagrams are obviously the same as the previous problem, but we must carefully consider the arrows. Using the Feynman Rules in problem 10.2, and choosing to let k_1 and k_2 be the a-type particles, we draw the arrows like this:



Which of these diagrams is s-channel? We could write a very formal argument, but instead let's just notice that in the previous problem, the s-channel is the only one that carried any momentum across the loop when the masses were on-shell. Hence, the first diagram is still the s-diagram.

What's different from the previous problem? The arrows and the symmetry factors. The arrows don't matter, because the swapped arrows also have swapped signs on the momenta (see the solution to problem 10.2), so this is inconsequential. The symmetry factors are different though: the t- and u-channel diagrams have a symmetry factor of one, and so the diagrams contribute twice as much as before. Using equations (16.1.3) and (16.1.4), then, we have:

$$iV_4 = -iZ_\lambda\lambda + \frac{i\lambda^2}{16\pi^2}\left[\frac{5}{\varepsilon} + 5ln\left(\frac{\mu}{m}\right) + 1\right] + O(\lambda^4)$$

which, after we absorb our divergences, infinities, and normalizing conventions into Z_λ, gives us:

$$C = \frac{5\lambda}{16\pi^2}\left[\frac{1}{\varepsilon} + ln\left(\frac{\mu}{m}\right) + \frac{1}{5}\right] + O(\lambda^2)$$

$$V_4 = -\lambda$$