Srednicki Chapter 15

QFT Problems & Solutions

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Srednicki 15.1. In this problem, we will verify the result of problem 13.1 to $\mathbf{O}(\boldsymbol{\alpha})$. (a) Let $\mathbf{\Pi_{loop}}$ be given by the first line of equation 14.32, with $\varepsilon > 0$. Show that, up to $\mathbf{O}(\boldsymbol{\alpha^2})$ corrections,

$$\mathbf{A} = \mathbf{\Pi'_{loop}}(-\mathbf{m^2})$$

Then use Cauchy's integral formula to write this as

$$\mathbf{A} = \oint \frac{\mathbf{dw}}{\mathbf{2\pi i}} \frac{\mathbf{\Pi_{loop}}(\mathbf{w})}{(\mathbf{w}+\mathbf{m^2})^2}$$

where the coutour of integration is a small counterclockwise circle around $-\mathbf{m^2}$ in the complex w plane.

Using equation 14.32, we have:

$$\Pi(k^2) = \Pi_{loop}(k^2) - Ak^2 - Bm^2 + O(\alpha^2)$$

which implies:

$$\Pi'(k^2) = \Pi'_{loop}(k^2) - A + O(\alpha^2)$$

Taking $k^2 = -m^2$, and neglecting second-order corrections:

$$\Pi'(-m^2) = \Pi'_{loop}(-m^2) - A$$

The left hand side is zero by the boundary condition (equation 14.8), so;

$$A = \Pi'_{loop}(-m^2)$$

as expected. Now recall that Cauchy's integral formula is that:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint \frac{f(w)}{(w-a)^{n+1}} dw$$

Taking $n = 1$ and $a = -m^2$, we have:

$$A = \Pi'(-m^2) = \frac{1}{2\pi i} \oint \frac{\Pi_{loop}(w)}{(w+m^2)^2} dw$$

also as expected.

(b) By examining equation 14.32, show that the only singularity of $\mathbf{\Pi_{loop}(k^2)}$ is a branch point at $\mathbf{k^2 = -4m^2}$. Take the cut to run along the negative real axis.

Let's remember what a branch point is. A branch point is when the function is discontinuous when evaluated around a circle of the complex plane. Therefore, 0 is a branch point of the natural logarithm. To see this, let's start at $z = Re^{i\theta}$, where R is arbitrarily small. Recall the definition of the complex log function: $ln\ z = ln\ r + i(\theta + 2k_\pi)$, where k_π is an integer. Let's say we start with $\theta = 0$, and so $ln(z) = ln(r)$. After one complete loop, we have $ln(z) = ln(r) + i(2\pi)$, which is not the same value. Hence there is a discontinuity, since $ln(Re^0) = lnR$, but $ln(Re^{0-i\epsilon}) = lnR + i(2\pi - \epsilon)$. More to the point, one loop puts us on a different branch (value of k_π), hence the name.

The same argument could be used for any negative point – for example at -3, we have $ln\ z = ln\ \sqrt{9 + 6\varepsilon cos\theta + \varepsilon^2} + i(\theta + 2k_\pi)$. One loop around will take us back to where we started with an additional factor of $i2\pi$ just as before, so the discontinuity is still there.

On the other hand, if D is real and positive, then we have the real natural log function, which is not multi-valued. Using the discussion in the last paragraph of chapter 15, we conclude that there is a branch cut whenever $k^2 < -4m^2$, and a branch point at the end, when $k^2 = -4m^2$. That accounts for every possible value of k, and so we've identified all the singularities (there may be singularities in terms of other variables, like ε, but we are not concerned with those since they are not being integrated over).

(c) Distort the contour in equation 15.15 to a circle at infinity with a detour around the branch cut. Examine equation 14.32 to show that, for $\boldsymbol{\varepsilon > 0}$, the circle at infinity does not contribute. The contour around the branch cut then yields:

$$\mathbf{A} = \int_{-\infty}^{-\mathbf{4m^2}} \frac{\mathbf{dw}}{\mathbf{2\pi i}} \frac{\mathbf{1}}{\mathbf{(w + m^2)^2}} \left[\mathbf{\Pi_{loop}(w + i\epsilon) - \Pi_{loop}(w - i\epsilon)}\right]$$

where $\boldsymbol{\epsilon}$ is infinitesimal (and is not to be confused with $\boldsymbol{\varepsilon = 6 - d}$).

We see that at large w, $A \sim \frac{\Pi_{loop}(w)}{w^2}$. Using equation 14.32, we have $A \sim |w|^{-1-\varepsilon/2}$, from which it is obvious that the contribution to A vanishes in the limit of large $|w|$.

(d) Examine equation 14.32 to show that the real part of $\mathbf{\Pi_{loop}(w)}$ is continuous across the branch cut, and that the imaginary part changes sign, so that

$$\mathbf{\Pi_{loop}(w + i\epsilon) - \Pi_{loop}(w - i\epsilon) = -2iIm\ \Pi_{loop}(w - i\epsilon)}$$

In equation 14.32, we have

$$D^{1-\varepsilon/2}$$

And so, above the branch cut we have:

$$(|D|e^{i(\pi-\epsilon)})^{1-\varepsilon/2}$$

Now take $\epsilon \to 0$, and we have

$$(|D|e^{i\pi})^{1-\varepsilon/2}$$

This gives:

$$|D|^{1-\varepsilon/2}e^{i\pi(1-\varepsilon/2)} = |D|^{1-\varepsilon/2}\left[cos(\pi-\varepsilon/2)+isin(\pi-\varepsilon/2)\right] \tag{15.1.1}$$

Below the branch cut we have:

$$(|D|e^{-i(\pi-\epsilon)})^{1-\varepsilon/2}$$

Taking $\epsilon \to 0$ and simplifying, remembering that cosine is even while sine is odd, we have:

$$|D|^{1-\varepsilon/2}e^{i\pi(1-\varepsilon/2)} = |D|^{1-\varepsilon/2}\left[cos(\pi-\varepsilon/2)-isin(\pi-\varepsilon/2)\right] \tag{15.1.2}$$

Comparing (15.1.1) and (15.1.2), we see that the real parts are equal and the imaginary parts are opposite, exactly as claimed.

(e) Let $\mathbf{w = -s}$ in equation 15.16, and use equation 15.17 to get

$$\mathbf{A = -\frac{1}{\pi}\int_{4m^2}^{\infty} ds \frac{Im\ \Pi_{loop}(-s-i\epsilon)}{(s-m^2)^2}}$$

Use this to verify the result of problem 13.1 to $\boldsymbol{O(\alpha)}$.

This equation follows directly from applying the result of part (d) to the result of part (c).

As for verifying the result from problem 13.1 – recall that our result from problem 13.1 was:

$$Z_\phi^{-1} = 1 + \int_{4m^2}^{\infty} ds\ \rho(s) = (1+A)^{-1}$$

Of course, $A^{-1} = 1 - A + O(\alpha^2)$, so:

$$A = -\int_{4m^2}^{\infty} ds\ \rho(s)$$

Now we use equation 15.13:

$$A = -\frac{1}{\pi}\int_{4m^2}^{\infty} ds\ \frac{Im\ \Pi(-s)}{(-s+m^2-Re\ \Pi(-s))^2+(Im\ \Pi(-s))^2}$$

Since $\Pi(-s)$ is of $O(\alpha)$ (or higher), squaring them does not contribute to the order at which we are working. Adding the factor of $i\epsilon$ is also allowed (ϵ by definition is negligable after all, so we can add it wherever we want. It's getting rid of ϵ that's the hard part!). This yields equation 15.18.

Srednicki 15.2. *Dispersion relations.* Consider the exact $\mathbf{\Pi(k^2)}$, with $\boldsymbol{\varepsilon = 0}$. Assume that its only singularity is a branch point at $\mathbf{k^2 = -4m^2}$, that it obeys

equation 15.17, and that $\mathbf{\Pi(k^2)}$ grows more slowly than $\mathbf{|k^2|^2}$ at large $\mathbf{|k^2|}$. By recapitulating the analysis in the previous problem, show that:

$$\mathbf{\Pi''(k^2) = \frac{2}{\pi}\int_{4m^2}^{\infty} ds \frac{Im\ \Pi(-s - i\varepsilon)}{(k^2 + s)^3}}$$

This is a *twice subtracted dispersion relation.* It gives $\mathbf{\Pi''(k^2)}$ throughout the complex $\mathbf{k^2}$ plane in terms of the values of the imaginary part of $\mathbf{\Pi(k^2)}$ along the branch cut.

We begin with Cauchy's Integral Formula:

$$\Pi''(k^2) = 2! \oint \frac{dw}{2\pi i} \frac{\Pi(w)}{(w - k^2)^3}$$

This is exactly equation 15.15 with $\Pi_{loop} \to 2\Pi(w)$, $m^2 \to -k^2$ and $2 \to 3$ in the denominator exponent. Taking these substitutions to equation 15.18 yields 15.19.