Srednicki Chapter 14

QFT Problems & Solutions

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Srednicki 14.1. Derive a generalization of Feynman's formula,

$$\frac{\mathbf{1}}{\mathbf{A_1^{\alpha_a}\ldots A_n^{\alpha_n}}} = \frac{\mathbf{\Gamma\left(\sum_i \alpha_i\right)}}{\mathbf{\prod_i \Gamma(\alpha_i)}}\frac{\mathbf{1}}{\mathbf{(n-1)!}}\int \mathbf{dF_n}\frac{\mathbf{\prod_i x_i^{\alpha_i-1}}}{\mathbf{\left(\sum_i x_i A_i\right)^{\sum_i \alpha_i}}}$$

Hint: start with:

$$\frac{\mathbf{\Gamma(\alpha)}}{\mathbf{A^\alpha}} = \int_{\mathbf{0}}^{\mathbf{\infty}} \mathbf{dt\ t^{\alpha-1}e^{-At}}$$

which defines the gamma function. Put an index on A, $\boldsymbol{\alpha}$, and t, and take the product. Then multiply on the right hand side by

$$\mathbf{1} = \int_{\mathbf{0}}^{\mathbf{\infty}} \mathbf{ds}\ \boldsymbol{\delta}\left(\mathbf{s} - \sum_{\mathbf{i}} \mathbf{t_i}\right)$$

Make the change of variable $\mathbf{t_i = sx_i}$, and carry out the integral over s.

Following the hint, we start with:

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty dt\ t^{\alpha-1}e^{-At}$$

Putting an index on A, α and t, and taking the product of both sides, we have:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^\infty dt_i\ t_i^{\alpha_i-1}e^{-A_i t_i}$$

Now we multiply on the right by equation 14.51:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^\infty dt_i ds\ t_i^{\alpha_i-1}e^{-A_i t_i}\delta\left(s - \sum_i t_i\right)$$

Next we make the change of variable $t_i = sx_i$:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^\infty dx_i ds\ (sx_i)^{\alpha_i-1}e^{-A_i s x_i}s\delta\left(s - \sum_i sx_i\right)$$

Recall that the delta function has the property of $f(cx) = \frac{1}{|c|}f(x)$. Since s is always positive, we have:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^\infty dx_i ds\ (sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \delta\left(1 - \sum_i x_i\right)$$

Now let's add in an identity:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^\infty dx_i ds\ (sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \frac{(n-1)!}{(n-1)!} \delta\left(1 - \sum_i x_i\right)$$

Next, note that all the x_i terms must be positive (thanks to our integral), but can't be greater than one (thanks to the delta function). So, we can rewrite the x integral:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int_0^1 dx_i \int_0^\infty ds\ (sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \frac{(n-1)!}{(n-1)!} \delta\left(1 - \sum_i x_i\right)$$

Using equation 14.10:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int dF_n \int_0^\infty ds\ (sx_i)^{\alpha_i - 1} e^{-A_i s x_i} \frac{1}{(n-1)!}$$

Separating out the s-integral:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \prod_i \int dF_n\ (x_i)^{\alpha_i - 1} \frac{1}{(n-1)!} \int_0^\infty ds s^{\alpha_i - 1} e^{-A_i s x_i}$$

This is quickly going to get confusing with the product. To avoid this, let's have the product act on each term separately. This, of course, is not always allowed, but it's fine in this case since everything is multiplied together.

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \int dF_n \left[\prod_i (x_i)^{\alpha_i - 1}\right] \frac{1}{(n-1)!} \int_0^\infty ds \left[\prod_i s^{\alpha_i - 1}\right] \left[\prod_i e^{-A_i s x_i}\right]$$

These last two terms give products of exponentials, which of course is a sum in the exponent. So:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \int dF_n \left[\prod_i (x_i)^{\alpha_i - 1}\right] \frac{1}{(n-1)!} \int_0^\infty ds \left[s^{\sum_i \alpha_i - 1}\right] \left[e^{-s \sum_i A_i x_i}\right]$$

We can solve this s-integral on Mathematica or the equivalent, the result is:

$$\prod_i \frac{\Gamma(\alpha_i)}{A_i^\alpha} = \int dF_n \left[\prod_i (x_i)^{\alpha_i - 1}\right] \frac{1}{(n-1)!} \Gamma\left[\sum_i \alpha_i\right] \left(\sum_i A_i x\right)^{-\sum_i \alpha_i}$$

Distributing the product on the left hand side, we have:

$$\frac{\prod_i \Gamma(\alpha_i)}{A_1^{\alpha_1} \dots A_n^{\alpha_n}} = \int dF_n \left[\prod_i (x_i)^{\alpha_i - 1}\right] \frac{1}{(n-1)!} \Gamma\left[\sum_i \alpha_i\right] \left(\sum_i A_i x\right)^{-\sum_i \alpha_i}$$

Then:

$$\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \frac{\Gamma\left(\sum_i \alpha_i\right)}{\prod_i \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \, \frac{\prod_i (x_i)^{\alpha_i - 1}}{\left(\sum_i A_i x\right)^{\sum_i \alpha_i}}$$

Srednicki 14.2. Verify equation 14.23.

Srednicki gave us the opening here when he suggested taking a Gaussian integral in cartesian and spherical coordinates. Let's do cartesian coordinates first (we'll also use x as our dummy variable):

$$\int_{-\infty}^{\infty} d^d x e^{-\left(\sum_i x_i^2\right)} = (\sqrt{\pi})^d$$

Next let's do it in spherical coordinates:

$$\int d\Omega \int_0^{\infty} dr r^{d-1} e^{-r^2} = \frac{\Omega}{2}\Gamma(d/2)$$

Equating these results, we have:

$$\Omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Srednicki 14.3. (a) Show that:

$$\int \mathbf{d^d q q^\mu f(q^2) = 0}$$

$$\int \mathbf{d^d q q^\mu q^\nu f(q^2) = C_2 g^{\mu\nu} \int d^d q q^2 f(q^2)}$$

and evaluate the constant $\mathbf{C_2}$ in terms of d. Hint: use Lorentz symmetry to argue for the general structure, and evaluate $\mathbf{C_2}$ by contracting with $\mathbf{g_{\mu\nu}}$.

The first equation is odd, and so integrates to zero.

The second equation has two indices, but cannot depend on any four-vectors. Hence, the general structure is, by Lorentz Symmetry:

$$\int d^d q \; q^\mu q^\nu f(q^2) = A g^{\mu\nu} \tag{14.3.1}$$

Contracting with $g_{\mu\nu}$, we find:

$$\int d^d q q^2 f(q^2) = Ad$$

and so:

$$A = \frac{1}{d} \int d^d q q^2 f(q^2)$$

Plugging this into equation (14.3.1) gives:

$$\int d^d q q^\mu q^\nu f(q^2) = \frac{1}{d} g^{\mu\nu} \int d^d q q^2 f(q^2)$$

Comparing this to equation 14.53, we find that $C_2 = 1/d$.

(b) Similarly evaluate $\int \mathbf{d^d q\ q^\mu q^\nu q^\rho q^\sigma f(q^2)}$.

As before, Lorentz Symmetry tells us that we must use the g. This time there are three ways to combine the indices:

$$\int d^d q\ q^\mu q^\nu q^\rho q^\sigma f(q^2) = A g^{\mu\nu} g^{\rho\sigma} + B g^{\mu\rho} g^{\nu\sigma} + C g^{\mu\sigma} g^{\rho\nu} \tag{14.3.2}$$

To evaluate A, B, and C, let's start by contracting with $g_{\mu\nu} g_{\rho\sigma}$

$$\int d^d q\ q^4 f(q^2) = Ad^2 + Bd + Cd$$

And now let's contract with $g_{\mu\rho} g_{\nu\sigma}$:

$$\int d^d q\ q^4 f(q^2) = Ad + Bd^2 + Cd$$

And now let's contract with $g_{\mu\sigma} g_{\rho\nu}$:

$$\int d^d q\ q^4 f(q^2) = Ad + Bd + Cd^2$$

Equating the three right hand sides, we find that $A = B = C$. Then, we have:

$$\int d^d q\ q^4 f(q^2) = A(2d + d^2)$$

Hence, we have:

$$A = B = C = \frac{1}{2d + d^2} \int d^d q\ q^4 f(q^2)$$

Putting this into equation (14.3.2), we have:

$$\int d^d q\ q^\mu q^\nu q^\rho q^\sigma f(q^2) = \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\rho\nu}}{2d + d^2} \int d^d q\ q^4 f(q^2)$$

Srednicki 14.4. Compute the values of $\boldsymbol{\kappa_A}$ and $\boldsymbol{\kappa_B}$.

Start with Srednicki 14.39:

$$\Pi(k^2) = \frac{1}{2}\alpha \int_0^1 dx D ln(D/m^2) + \alpha \left(\frac{1}{6}\kappa_A k^2 + \kappa_B m^2 \right) + O(\alpha^2)$$

Imposing the first boundary condition, we have:

$$\Pi(-m^2) = \frac{\alpha}{2} m^2 \int_0^1 dx (1 - x + x^2) ln(1 - x + x^2) - \frac{\alpha}{6}\kappa_A m^2 + \alpha \kappa_B m^2 = 0$$

Evaluating the integral:

$$\frac{1}{36}\left(3\sqrt{3}\pi - 19\right) - \frac{1}{6}\kappa_A + \kappa_B = 0 \tag{14.4.1}$$

Now let's take the derivative of equation 14.39 so that we can impose the second boundary condition:

$$\Pi'(k^2) = \frac{\alpha}{2}\int_0^1 dx x(1-x)\left[1 + ln\left(\frac{x(1-x)k^2 + m^2}{m^2}\right)\right] + \frac{\alpha}{6}\kappa_A$$

Imposing the second boundary condition:

$$\Pi'(-m^2) = \int_0^1 dx x(1-x)\left[1 + ln(1 - x + x^2)\right] + \frac{1}{3}\kappa_A = 0$$

Solving this for κ_A:

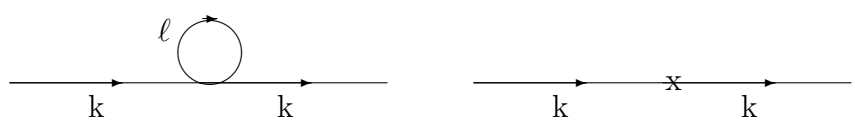
$$\kappa_A = \frac{7}{3} - \frac{\sqrt{3}\pi}{2} = -0.3874$$

Using this in equation (14.4.1), we have:

$$\kappa_B = \frac{11}{12} - \frac{\sqrt{3}\pi}{6} = .0098$$

Srednicki 14.5. Compute the O(λ) corrections to the propagator in ϕ^4 theory (see problem 9.2), and compute the O(λ) terms in A and B.

We essentially have to repeat the entire analysis of this chapter for the ϕ^4 theory. Let's start by drawing the $O(\lambda)$ corrections to the vertex. They are:



Note that this second diagram is necessary because it may contribute at O(λ). Srednicki told us (and proved in equations 14.37, 14.38) that this diagram is of O(g^2) in ϕ^3 theory. We might assume that this diagram is similarly of O(λ^2) in ϕ^4 theory, but there's no way to know for sure – we'll have to assume for the moment that it does correct the propagator at O(λ). We will have calculated A and B by the end of the problem, so then we'll know whether it contributed anything or not. If not, then no harm is done, we'll just state that the diagram doesn't contribute and ignore all higher-order corrections, including those from this diagram.

Our first step is to write down the self-energy. This will be the values of these Feynman diagrams, remembering to ignore the external propagators. Following the Feynman rules for ϕ^4 theory, the first diagram contributes the following terms to the self-energy:

- -iλ for the vertex
- $-i/(\ell^2 + m^2 - i\epsilon)$ for the internal propagator of the loop.

- $\int d^d\ell/(2\pi)^d$ to integrate over the loop momentum.
- 1/2 for the symmetry factor of the loop
- $-i$ because the stated Feynman Rules are for $i\tau$ (or $i\Pi$, as the case may be)

The second diagram contributes only the counterterm vertex factor, which is the same as in ϕ^3 theory, since the counterterm Lagrangian is the same. We also put in the factor of $-i$, for the same reason as before. Putting all this together, we have:

$$\Pi(k^2) = (-i)\frac{(-i\lambda)}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{-i}{\ell^2+m^2-i\epsilon} + (-i)(-i)(Ak^2+Bm^2)$$

Writing this more neatly, we have:

$$\Pi(k^2) = \frac{i\lambda}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2-i\epsilon} + i(Ak^2+Bm^2)$$

It's convenient to write this as:

$$i\Pi(k^2) = -\frac{\lambda}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2-i\epsilon} - Ak^2 - Bm^2$$

The main challenge is to calculate this integral. Srednicki presented a bag of tricks, but the only one we need in this problem is the Wick Rotation. We define $\ell^0 = i\bar{\ell}_d$ and $\ell_j = \bar{\ell}_j$. Then $\ell^2 = \bar{\ell}^2$, but $d^d\ell = id^d\bar{\ell}$. The point of this, of course, is that we can get rid of the ϵ in the denominator, per equation 14.16. Then,

$$i\Pi(k^2) = -i\frac{\lambda}{2}\int \frac{d^d\bar{\ell}}{(2\pi)^d}\frac{1}{\bar{\ell}^2+m^2} - Ak^2 - Bm^2$$

We know how to solve this integral, but it's worth pausing to consider the dimensionality. This integral appears to be well defined only for d = 1 (or lower). However, using Srednicki's "trick #3," we could take two derivatives with respect to m^2. This would be allowed because we have two degrees of freedom, so we could then integrate twice, solve the boundary conditions, and return to the original problem. The advantage of this is that after differentiation, we would have an ℓ^6 in the denominator, and so the integral will converge so long as $d < 6$. As it turns out, we get a dimensionless coupling constant for $d = 4$, so this theory is renormalizable in the case of interest. Just as in the text, it is not necessary to actually take the two derivatives, but it is necessary to know that the theory is renormalizable, and that it is therefore valid to analytically continue our results to $d = 4$.

Now we can use equation 14.27:

$$\Pi(k^2) = -\frac{\lambda}{2}\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}}(m^2)^{-1+d/2} - Ak^2 - Bm^2$$

Now it's time to define $\varepsilon = 4 - d$ (4 because we get a dimensionless coupling constant when d = 4). Let's also set $\lambda \to \tilde{\mu}^\varepsilon\lambda$, such that all mass dimensionality is projected onto the $\tilde{\mu}$. This gives:

$$\Pi(k^2) = -\frac{\lambda\tilde{\mu}^\varepsilon}{2}\frac{\Gamma\left(-1+\frac{\varepsilon}{2}\right)}{(4\pi)^2}(4\pi)^{\varepsilon/2}(m^2)(m^2)^{-\varepsilon/2} - Ak^2 - Bm^2$$

which is:

$$\Pi(k^2) = -\frac{m^2\lambda}{2(4\pi)^2}\Gamma\left(-1+\frac{\varepsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)^{\varepsilon/2} - Ak^2 - Bm^2$$

Using equation 14.26 and 14.33, we simplify:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + 1\right]\left[1 + \frac{\varepsilon}{2}ln\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

Now multiply together, and keep only those terms of O(1) in ε, since we want to work in four dimensions. Then:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + 1 + ln\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

Next, let's solve equation 14.35, to find that $4\pi\tilde{\mu}^2 = e^\gamma\mu^2$. Then:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + 1 + ln\left(\frac{e^\gamma\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

which is:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + 1 + ln(e^\gamma) + ln\left(\frac{\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

and so:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + ln\left(\frac{\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

By the way, this is equation 31.5, so we know that we did well.

The problem is that we need $\Pi(k^2)$ to be finite and independent of μ. In terms of k^2, everything is fine at this order, so we'll take $A = \kappa_A\lambda + O(\lambda^2)$, where κ_A is purely numeric. Since we're ignoring $O(\lambda^2)$ corrections, we have:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + ln\left(\frac{\mu^2}{m^2}\right)\right] - \kappa_A\lambda k^2 - Bm^2$$

Now let's impose the boundary condition $\Pi'(-m^2) = 0$ where the prime represents a derivative with respect to k^2. This gives $-\kappa_A\lambda = 0$. Hence, $\kappa_A = 0$, and so $\boxed{A = O(\lambda^2)}$. This is because A clearly does not contribute at lowest order (otherwise the free propagator would have this counterterm vertex in it), and we've just shown that it does not contribute at first order. The self-energy is now:

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + ln\left(\frac{\mu^2}{m^2}\right)\right] - Bm^2$$

Now for B. This is neither finite nor independent of μ, so we will choose:

$$B = \frac{\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + ln\left(\frac{\mu^2}{m^2}\right)\right] + \lambda\kappa_B + O(\lambda^2)$$

which is

$$B = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \frac{1}{2} + ln\left(\frac{\mu}{m}\right)\right] + \lambda\kappa_B + O(\lambda^2)$$

Now we have:

$$\Pi(k^2) = -\lambda\kappa_B m^2 + O(\lambda^2)$$

Imposing the other boundary condition, $\Pi(-m^2) = 0$, we find that $\kappa_B = 0$. Then,

$$\boxed{B = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \frac{1}{2} + ln\left(\frac{\mu}{m}\right)\right] + O(\lambda^2)}$$

and

$$\boxed{\Pi(k^2) = O(\lambda^2)}$$

So the propagator is uncorrected to first order in λ. Further, we found that the counterterm vertex does contribute at $O(\lambda)$.

Srednicki 14.6. Repeat problem 14.5 for the theory of problem 9.3.

Fortunately, this theory is nearly identical to that of the previous problem, the only difference is the symmetry factor. Thanks to the complex field in problem 9.3, S = 1, and so there is an extra factor of 2 in the first diagram that propagates through the problem. The results are:

$$A = O(\lambda^2)$$

$$B = \frac{\lambda}{(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + ln\left(\frac{\mu^2}{m^2}\right)\right] + O(\lambda^2)$$

$$\Pi(k^2) = O(\lambda^2)$$

Srednicki 14.7. *Renormalization of the anharmonic oscillator.* Consider an anharmonic oscillator, specified by the Lagrangian

$$\mathbf{L = \frac{1}{2}Z\dot{q}^2 - \frac{1}{2}Z_\omega\omega^2 q^2 - Z_\lambda\lambda\omega^3 q^4}$$

We set $\hbar = 1$ and $m = 1$; λ is then dimensionless.

(a) Find the Hamiltonian H corresponding to L. Write it as $H = H_0 + H_1$, where $H_0 = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2$, and $[Q, P] = i$.

We take the conjugate momentum:

$$P = \frac{\partial L}{\partial \dot{q}} = Z\dot{q}$$

Then, the Hamiltonian is given by:

$$H = P\dot{q} - L$$

$$H = \frac{1}{2}Z\dot{q}^2 + \frac{1}{2}Z_\omega\omega^2 q^2 + Z_\lambda\lambda\omega^3 q^4$$

which is, taking $Q = q$ and P of course is the conjugate momentum as defined above,

$$H = \frac{1}{2}Z^{-1}P^2 + \frac{1}{2}Z_\omega\omega^2 Q^2 + Z_\lambda\lambda\omega^3 Q^4$$

Then:

$$H = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2 + \frac{1}{2}(Z^{-1} - 1)P^2 + \frac{1}{2}(Z_\omega - 1)\omega^2 Q^2 + Z_\lambda\lambda\omega^3 Q^4$$

where the first two terms are H_0 and the remainder are H_1. Choosing P to be the conjugate momentum of Q means that the commutation relation required by the problem is automatically satisfied.

(b) Let $|0\rangle$ and $|1\rangle$ be the ground and first excited states of $\mathbf{H_0}$, and let $|\Omega\rangle$ and $|\mathbf{I}\rangle$ be the ground and first excited states of H. We take all these eigenstates to have unit norm. We define $\boldsymbol{\omega}$ to be the excitation energy of $\boldsymbol{H}$, $\boldsymbol{\omega} = \mathbf{E_I} - \mathbf{E_\Omega}$. We normalize the position operator Q by setting $\langle \mathbf{I}|\mathbf{Q}|\boldsymbol{\Omega}\rangle = \langle \mathbf{1}|\mathbf{Q}|\mathbf{0}\rangle = (\mathbf{2}\boldsymbol{\omega})^{-1/2}$. Finally, to make things mathematically simpler, we set $\mathbf{Z}_{\boldsymbol{\lambda}}$ equal to one, rather than using a more physically motivated definition. Write $\mathbf{Z} = \mathbf{1} + \mathbf{A}$ and $\mathbf{Z}_{\boldsymbol{\omega}} = \mathbf{1} + \mathbf{B}$, where $\mathbf{A} = \boldsymbol{\kappa}_\mathbf{A}\boldsymbol{\lambda} + \mathbf{O}(\boldsymbol{\lambda}^2)$ and $\mathbf{B} = \boldsymbol{\kappa}_\mathbf{B}\boldsymbol{\lambda} + \mathbf{O}(\boldsymbol{\lambda}^2)$. Use Rayleigh-Schrödinger perturbation theory to compute the $\mathbf{O}(\boldsymbol{\lambda})$ corrections to the unperturbed energy eigenvalues and eigenstates.

This is just quantum mechanics. The energy of the unperturbed ith state is given by $E_i = \frac{(2i+1)\omega}{2}$. The first-order correction to this is, by Shankar 17.1.7:

$$E_i^1 = \langle i|H_1|i\rangle$$

which is:

$$E_i^1 = \langle i|\frac{1}{2}(Z^{-1} - 1)P^2 + \frac{1}{2}(Z_\omega - 1)\omega^2 Q^2 + Z_\lambda\lambda\omega^3 Q^4|i\rangle$$

Next we set $Z_\lambda = 1$, $Z = 1 + A$, and $Z_\omega = 1 + B$. Then,

$$E_i^1 = \langle i| - \frac{1}{2}AP^2 + \frac{1}{2}B\omega^2 Q^2 + \lambda\omega^3 Q^4|i\rangle$$

Splitting this up, and using the expansion of A and B, we have:

$$E_i^1 = -\frac{1}{2}\kappa_A\lambda\langle i|P^2|i\rangle + \frac{1}{2}\kappa_B\lambda\omega^2\langle i|Q^2|i\rangle + \lambda\omega^3\langle i|Q^4|i\rangle$$

Now we quantize the operators in the usual way, see Shankar 7.4.28 and 7.4.29, which state that:

$$Q = \frac{1}{\sqrt{2\omega}}(a^\dagger + a)$$

$$P = i\sqrt{\frac{\omega}{2}}(a^\dagger - a)$$

This scheme is great because the position operator is normalized "out of the box" to $(2\omega)^{-1/2}$ (between $\langle 1|$ and $|0\rangle$) as Srednicki prescribed. Thus our correction to the energy eigenvalues are:

$$E_i^1 = \frac{\omega\lambda}{4}\kappa_A\langle i|(a^\dagger - a)^2|i\rangle + \frac{\omega\lambda}{4}\kappa_B\langle i|(a+a^\dagger)^2|i\rangle + \frac{\lambda\omega}{4}\langle i|(a+a^\dagger)^4|i\rangle$$

which is:

$$E_i^1 = -\frac{\omega\lambda}{4}\kappa_A\langle i|aa^\dagger + a^\dagger a|i\rangle + \frac{\omega\lambda}{4}\kappa_B\langle i|aa^\dagger + a^\dagger a|i\rangle + \frac{\lambda\omega}{4}\langle i|aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + aa^\dagger a^\dagger a + a^\dagger aaa^\dagger$$

$$+a^\dagger aa^\dagger a + a^\dagger a^\dagger aa|i\rangle$$

Now let's simplify to the case where i = 0:

$$E_0^1 = -\frac{\omega\lambda}{4}\kappa_A + \frac{\omega\lambda}{4}\kappa_B + \frac{3\lambda\omega}{4}$$

Hence, the ground state energy of the perturbed system is:

$$\boxed{E_\Omega = \frac{\omega}{2} + \frac{\lambda\omega}{4}\left(-\kappa_A + \kappa_B + 3\right) + O(\lambda^2)}$$

Now for the case where i = 1:

$$E_1^1 = -\frac{3\omega\lambda}{4}\kappa_A + \frac{3\omega\lambda}{4}\kappa_B + \frac{\lambda\omega}{4}(6+4+2+2+1+0)$$

Hence, the first excited state energy of the perturbed system is

$$\boxed{E_I = \frac{3\omega}{2} + \frac{3\omega\lambda}{4}\left(-\kappa_A + \kappa_B + 5\right)}$$

Now for the corrections to the eigenstates, given by Shankar 17.1.14:

$$|n\rangle = |n^0\rangle + \sum_{n\neq m}\frac{\langle m|H^1|0\rangle}{E_0^0 - E_m^0}|m\rangle$$

Thus:

$$|\Omega\rangle = |0\rangle + \sum_{n\neq m}\frac{1}{\frac{\omega}{2} - \frac{(2m+1)\omega}{2}}\left[-\frac{1}{2}\kappa_A\lambda\langle m|P^2|0\rangle + \frac{1}{2}\kappa_B\lambda\omega^2\langle m|Q^2|0\rangle + \lambda\omega^3\langle m|Q^4|0\rangle\right]|m\rangle$$

which simplifies to:

$$|\Omega\rangle = |0\rangle + \sum_{n\neq m}\frac{\lambda}{4m}\left[-\kappa_A\langle m|(a^\dagger - a)^2|0\rangle - \kappa_B\langle m|(a^\dagger + a)^2|0\rangle - \langle m|(a^\dagger + a)^4|0\rangle\right]|m\rangle$$

In the first two bra-kets, $(a^\dagger \pm a)^2 = a^\dagger a^\dagger \pm aa^\dagger \pm a^\dagger a + aa$. These last two terms annihilate the vacuum, giving a zero. The second term gives a zero also since $m \neq n$. Then:

$$|\Omega\rangle = |0\rangle + \sum_{n\neq m}\frac{\lambda}{4m}\left[-\kappa_A\langle m|a^\dagger a^\dagger|0\rangle - \kappa_B\langle m|a^\dagger a^\dagger|0\rangle - \langle m|(a^\dagger + a)^4|0\rangle\right]|m\rangle$$

These first two terms are only defined if $m = 2$. Then,

$$|\Omega\rangle = |0\rangle + \frac{\lambda}{8}\left[-\kappa_A\langle 2|a^\dagger a^\dagger|0\rangle - \kappa_B\langle 2|a^\dagger a^\dagger|0\rangle\right]|m\rangle - \sum_{n\neq m}\frac{\lambda}{4m}\langle m|(a^\dagger + a)^4|0\rangle|m\rangle$$

Simplifying:

$$|\Omega\rangle = |0\rangle - \frac{\sqrt{2}\lambda(\kappa_A + \kappa_B)}{8}|2\rangle - \sum_{n\neq m}\frac{\lambda}{4m}\langle m|(a^\dagger + a)^4|0\rangle|m\rangle$$

Doing the expansions as before, we find that the last bra-ket gives $\sqrt{24}$ if m = 4, and $6\sqrt{2}$ if m = 2. Then,

$$|\Omega\rangle = |0\rangle - \frac{\sqrt{2}\lambda(\kappa_A + \kappa_B)}{8}|2\rangle - \frac{3\lambda\sqrt{2}}{4}|2\rangle - \frac{\sqrt{6}\lambda}{8}|4\rangle$$

This gives:

$$\boxed{|\Omega\rangle = |0\rangle - \frac{\sqrt{2}\lambda(\kappa_A + \kappa_B + 6)}{8}|2\rangle - \frac{\sqrt{6}\lambda}{8}|4\rangle + O(\lambda^2)}$$

We do the same procedure for $|I\rangle$, I'll skip the gory details. The result is:

$$\boxed{|I\rangle = |1\rangle - \frac{\lambda\sqrt{6}(\kappa_A + \kappa_B + 10)}{8}|3\rangle - \frac{\lambda\sqrt{30}}{8}|5\rangle + O(\lambda^2)}$$

(c) Find the numerical values of $\boldsymbol{\kappa_A}$ and $\boldsymbol{\kappa_B}$ that yield $\boldsymbol{\omega = E_I - E_\Omega}$ and $\boldsymbol{\langle I|Q|\Omega\rangle = (2\omega)^{-1/2}}$.

Imposing the first condition, we have:

$$E_I - E_\Omega = \omega + \frac{\omega\lambda}{2}(-\kappa_A + \kappa_B + 6) = \omega$$

This implies that:

$$\kappa_A - \kappa_B = 6 \tag{14.7.1}$$

Imposing the second condition, we have:

$$\langle I|Q|\Omega\rangle = \frac{1}{\sqrt{2\omega}}\left[\langle 1| - \frac{\lambda\sqrt{6}(\kappa_A + \kappa_B + 10)}{8}\langle 3| - \frac{\lambda\sqrt{30}}{8}\langle 5|\right]$$

$$\left(a^\dagger + a\right)\left[|0\rangle - \frac{\sqrt{2}\lambda(\kappa_A + \kappa_B + 6)}{8}|2\rangle - \frac{\sqrt{6}\lambda}{8}|4\rangle\right] = \frac{1}{\sqrt{2\omega}}$$

Using the operators and equation (14.7.1), we have:

$$\langle I|Q|\Omega\rangle = \frac{1}{\sqrt{2\omega}}\left[\langle 1| - \frac{\lambda\sqrt{6}(\kappa_A + 2)}{4}\langle 3| - \frac{\lambda\sqrt{30}}{8}\langle 5|\right] \times$$

$$\left[|1\rangle - \frac{\sqrt{6}\lambda\kappa_A}{4}|3\rangle - \frac{\lambda\kappa_A}{2}|1\rangle - \frac{\sqrt{30}\lambda}{8}|5\rangle - \frac{\sqrt{6}\lambda}{4}|3\rangle\right]$$

This gives:

$$\langle I|Q|\Omega\rangle = \frac{1}{\sqrt{2\omega}}\left[1 - \frac{\lambda\kappa_A}{2} + O(\lambda^2)\right]$$

Hence, $\boxed{\kappa_A = 0}$. Using equation (14.7.1), we have then that $\boxed{\kappa_B = -6}$.

(d) Now think of the Lagrangian in equation 14.54 as specifying a quantum field theory in d = 1 dimensions. Compute the O($\boldsymbol{\lambda}$) correction to the propagator. Fix $\boldsymbol{\kappa_A}$ and $\boldsymbol{\kappa_B}$ by requiring the propagator to have a pole at $\mathbf{k^2 = -\omega^2}$ with residue one. Do your results agree with those of part (c)? Should they?

Let's start by writing the Lagrangian again:

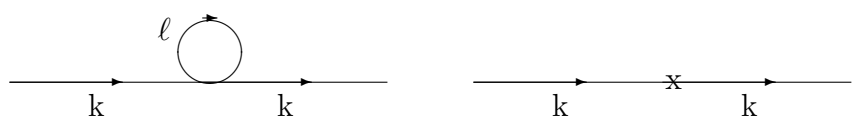
$$L = \frac{1}{2}Z\dot{q}^2 - \frac{1}{2}Z_\omega\omega^2 q^2 - Z_\lambda\lambda\omega^3 q^4$$

Now we'll write this in the usual way:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{1}{2}(Z-1)\dot{q}^2 - \frac{1}{2}(Z_\omega - 1)\omega^2 q^2 - Z_\lambda\lambda\omega^3 q^4$$

where the first two terms are the free field terms, the second two terms are the counterterms, and the final term is the interaction.

Next, let's draw the terms that would correct the propagator at $O(\lambda)$:



This is a new theory, so we have to use new Feynman rules. Fortunately these are easy to derive from the Lagrangian. For the first term, we have the vertex factor, the internal propagator, the symmetry factor, and the integral over ℓ. Then,

$$i\Pi_1(k^2) = \frac{1}{2}\int \frac{24\lambda\omega^3}{\ell^2 + \omega^2 - i\varepsilon}\frac{d\ell}{2\pi}$$

where the symmetry factor is 2 since exchanging the two ends of the propagator is equivalent to exchanging those two vertices, the vertex factor includes a factor of 24 since the Qs can be paired with the vertices in any of 4! different ways, and the integral is one dimensional since the problem tells us to work oin one dimension. By the way, if the factor of 24 seems unusual, that's because most Lagrangians have cleverly-chosen coefficients to cancel these factors, see for example the Lagrangian of problem 9.2. Note also that we have ω rather than m, since ω has taken the place of mass in our problem.

Next we do a Wick rotation, the result is:

$$\Pi_1(k^2) = \frac{6\lambda\omega^3}{\pi}\int \frac{1}{\ell^2 + \omega^2}d\ell$$

This integral is simple to evaluate, using equation 14.27 or other methods. We find:

$$\Pi_1(k^2) = 6\lambda\omega^2$$

That seems sufficiently simplified! Let's turn to the contribution from the second diagram. The only thing we have to worry about is the vertex factor, which we read off from the counterterm Lagrangian. The result is:

$$\Pi_2(k^2) = -\left((Z-1)k^2 - (Z_\omega - 1)\omega^2\right)$$

This should not be unclear, but if it is, it can also be obtained by comparing our Lagrangian with equation 9.9, and making the necessary substitutions to modify Feynman Rule #9 on page 77.

At any rate, we now use the information in part (b), $Z - 1 = \kappa_A\lambda$ and $Z_\omega - 1 = \kappa_B\lambda$. Then,

$$\Pi_2(k^2) = -\lambda\left(\kappa_A k^2 - \kappa_B\omega^2\right)$$

The entire self-energy (at $O(\lambda)$) is then:

$$\Pi(k^2) = 6\lambda\omega^2 - \lambda\left(\kappa_A k^2 - \kappa_B\omega^2\right)$$

The propagator will have a pole at $k^2 = -m^2$ with residue one if the boundary condition 14.7 and 14.8 are met. Imposing 14.8, we find:

$$\Pi'(k^2) = -\lambda\kappa_A = 0$$

Hence $\kappa_A = 0$. Imposing 14.7, we have:

$$\Pi(-m^2) = 6\lambda\omega^2 + \lambda\kappa_B\omega^2 = 0$$

which implies that $\kappa_B = -6$, in agreement with the result from part (c).

Finally, we write the correction to the propagator:

$$\boxed{\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + \omega^2 - i\varepsilon + O(\lambda^2)}}$$

Finally, we have the question of whether the agreement between (c) and (d) is expected. Our method of correcting the propagator in part (d) involves using our Feynman Rules, which we derived from the LSZ formula, which depends on the Klein-Gordon Equation. It may seem as though this is therefore incompatible with quantum mechanics. On the other hand, perturbation theory is not strictly limited to quantum mechanics: a close look at the derivation of perturbation theory (see for example Griffiths 6.1) shows that there is no dependence on the Schrödinger equation or any other tenets that violate field theory. In fact, the bra-kets that we evaluated in part (b) are dependent only on the definition of ω and Q, and both of those are defined in a way which is consistent with field theory. Hence, we do expect these to agree.

Note: In Srednicki's solutions, his equation 14.68 has the order of the terms in the denominator reversed. This is easily verified, see for example Shankar 17.1.13, Sakurai 5.1.44, or Griffiths 6.13. Fortunately, this just results in a negative sign that cancels out in the middle, so his conclusions are correct, though some terms in his corrected wave-functions in part (b) are off by a sign.