Srednicki Chapter 13

QFT Problems & Solutions

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Srednicki 13.1. Consider an interacting scalar field theory in d spacetime dimensions,

$$\mathcal{L} = -\frac{1}{2}\mathbf{Z}_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}\mathbf{Z_m m^2}\phi^2 - \mathcal{L}_1(\phi)$$

where $\mathcal{L}_1(\phi)$ is a function of ϕ (and not its derivatives). The exact momentum-space propagator for ϕ can be expressed in Lehmann-Källén form by equation 13.17. Find a formula for the renormalizing factor Z_ϕ in terms of ρ(s). Hint: consider the commutator $[\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})]$.

Fortunately we have a hint, otherwise this problem might seem baffling. Following the hint, we'll start by evaluating $\langle 0|\phi(x)\dot{\phi}(y)|0\rangle$, using equation 13.13.

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \widetilde{dk} e^{ik(x-y)} + \int_{4m^2}^{\infty} ds\ \rho(s) \int \widetilde{dk} e^{ik(x-y)}$$

which implies:

$$\langle 0|\phi(x)\dot{\phi}(y)|0\rangle = i\int \widetilde{dk} k^0 e^{ik(x-y)} + i\int_{4m^2}^{\infty} ds\ \rho(s) \int \widetilde{dk} k^0 e^{ik(x-y)}$$

Expanding the differential, we have:

$$\langle 0|\phi(x)\dot{\phi}(y)|0\rangle = i\int \frac{d^d k}{(2\pi)^d 2} e^{ik(x-y)} + i\int_{4m^2}^{\infty} ds\ \rho(s) \int \frac{d^d k}{(2\pi)^3 2} e^{ik(x-y)}$$

Next, let's set $x^0 = y^0$, ie take the commutator at equal times:

$$\langle 0|\phi(x)\dot{\phi}(y)|0\rangle = \frac{i}{(2\pi)^d 2}\left[\int d^d k e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + \int_{4m^2}^{\infty} ds\ \rho(s) \int d^d k e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\right]$$

These k-integrals are delta functions:

$$\langle 0|\phi(x)\dot{\phi}(y)|0\rangle = \frac{i}{2}\left[\delta^d(\vec{x}-\vec{y}) + \int_{4m^2}^{\infty} ds\ \rho(s)\delta^d(\vec{x}-\vec{y})\right]$$

Simplifying:

$$\langle 0|\phi(x)\dot{\phi}(y)|0\rangle = \frac{i\delta^d(\vec{x}-\vec{y})}{2}\left[1 + \int_{4m^2}^{\infty} ds\ \rho(s)\right]$$

If we do the same thing with the position of x and y swapped, we get the same thing with a minus sign (the y in the exponent is now positive, so the time-component is negative). Hence, the commutator is:

$$\langle 0|[\phi(x), \dot{\phi}(y)]|0\rangle = i\delta^d(\vec{x}-\vec{y})\left[1+\int_{4m^2}^{\infty} ds\rho(s)\right] \tag{13.1.1}$$

Next, let's try to find a conjugate variable for ϕ. Recall by definition that:

$$\Pi(y) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}Z_\phi \partial^0\phi\partial_0\phi - \frac{1}{2}Z_\phi\partial^i\phi\partial_i\phi - \frac{1}{2}Z_m m^2\phi^2 - \mathcal{L}_1(\phi)\right)$$

where we use the usual convention of Greek indices to sum over all time-space dimensions and Latin indices to sum over spatial indices only. Hence,

$$\Pi(y) = Z_\phi \dot{\phi} \tag{13.1.2}$$

Substituting (13.1.2) into (13.1.1), we find:

$$\langle 0|[\phi(x), \Pi(y)]|0\rangle = iZ_\phi\delta^d(\vec{x}-\vec{y})\left[1+\int_{4m^2}^{\infty} ds\ \rho(s)\right]$$

Now we use equation 3.28 on the left side (generalizing to d dimensions):

$$\langle 0|i\delta^d(\vec{x}-\vec{y})|0\rangle = iZ_\phi\delta^d(\vec{x}-\vec{y})\left[1+\int_{4m^2}^{\infty} ds\ \rho(s)\right]$$

These have no effect on the bra-ket, which then vanishes (because $\langle 0|0\rangle = 1$). Then:

$$i\delta^d(\vec{x}-\vec{y}) = iZ_\phi\delta^d(\vec{x}-\vec{y})\left[1+\int_{4m^2}^{\infty} ds\ \rho(s)\right]$$

from which we conclude:

$$Z_\phi = \frac{1}{1+\int_{4m^2}^{\infty} ds\ \rho(s)}$$