Srednicki Chapter 11

QFT Problems & Solutions

A. George

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Srednicki 11.1. (a) Consider a theory of two real scalar fields A and B with an interaction $\mathcal{L}_1 = gAB^2$. Assuming that $m_A > 2m_B$, compute the total decay rate of the A particle at tree level.

Equation 11.49 is:

$$\Gamma = \frac{1}{S}\int d\Gamma$$

Using equation 11.48:

$$\Gamma = \frac{1}{S}\int \frac{1}{2E_1}|\tau|^2 dLIPS_2(k_1)$$

Using equation 11.30:

$$\Gamma = \frac{1}{S}\int \frac{1}{2E_1}|\tau|^2 \frac{|\vec{k}_1'|}{16\pi^2\sqrt{s}} d\Omega_{CM}$$

Now using equation 11.2:

$$\Gamma = \frac{1}{S}\int \frac{1}{2E_1}|\tau|^2 \frac{\frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2}}{16\pi^2\sqrt{s}} d\Omega_{CM}$$

Now remember that for decay, $s = m_1^2 = E_1^2$. So we can simplify,

$$\Gamma = \frac{1}{S}\int \frac{1}{2m_1}|\tau|^2 \frac{\frac{1}{2\sqrt{m_1^2}}\sqrt{m_1^4 - 2(m_{1'}^2 + m_{2'}^2)m_1^2 + (m_{1'}^2 - m_{2'}^2)^2}}{16\pi^2 m_1} d\Omega$$

Cleaning up:

$$\Gamma = \frac{1}{S}\frac{1}{64\pi^2 m_1^3}\int |\tau|^2 \sqrt{m_1^4 - 2(m_{1'}^2 + m_{2'}^2)m_1^2 + (m_{1'}^2 - m_{2'}^2)^2} d\Omega \qquad (11.1.1)$$

where we dropped the CM subscript since everything is either taken in the CM frame or frame-invariant. Notice that this is a totally general result for one scalar decaying into two scalars, and we may wish to use this later.

In any case, we're presently presented with two identical scalars, so $m_{1'} = m_{2'} = m_B$. Then:

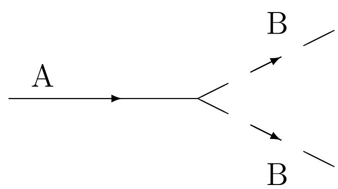
$$\Gamma = \frac{1}{S}\frac{1}{64\pi^2 m_A^3}\int |\tau|^2 \sqrt{m_A^4 - 4m_B^2 m_A^2}\, d\Omega$$

which is:

$$\Gamma = \frac{1}{S}\frac{1}{64\pi^2 m_A}\sqrt{1 - 4\frac{m_B^2}{m_A^2}}\int |\tau|^2 d\Omega \qquad (11.1.2)$$

Again, this result is valid for any one scalar decaying into two identical scalars, so we may wish to use this later.

Now we need the matrix element and the symmetry factors, both of which require the diagram. The only diagram for this process is:



Clearly, this diagram's matrix element is only the vertex factor, since the external lines add only a factor of 1. Remember how we construct the vertex factor (here I'll use the practical rules outlined in the ch. 10 slides):

1. Turn derivatives into ik^μ. There are no derivatives, so this doesn't matter.
2. Rub out the fields. This leaves us with g.
3. Add a factor of i. This leaves us with ig.
4. Multiply through by symmetry. In this case, swapping the B vertices is the same as swapping the propagators themselves, so S = 2, and the vertex factor is $2ig$.

With this, the decay rate is given by:

$$\Gamma = \frac{g^2}{32\pi^2 m_A}\sqrt{1 - 4\frac{m_B^2}{m_A^2}}\int d\Omega$$

The integral is obviously 4π. Thus,

$$\Gamma = \frac{g^2}{8\pi m_A}\sqrt{1 - 4\frac{m_B^2}{m_A^2}}$$

(b) Consider a theory of a real scalar field ϕ and a complex scalar field χ with $\mathcal{L}_1 = g\phi\chi^\dagger\chi$. Assuming that $m_\phi > 2m_\chi$, compute the total decay rate of the ϕ particle at tree level.

Now we go back to equation (11.1.2):

$$\Gamma = \frac{1}{S}\frac{1}{64\pi^2 m_\phi}\sqrt{1-4\frac{m_\chi^2}{m_\phi^2}}\int |\tau|^2 d\Omega$$

This time the vertex factor is ig and the symmetry factor is one. Then,

$$\Gamma = \frac{g^2}{64\pi^2 m_\phi}\sqrt{1-4\frac{m_\chi^2}{m_\phi^2}}\int d\Omega$$

which gives:

$$\Gamma = \frac{g^2}{16\pi m_\phi}\sqrt{1-4\frac{m_\chi^2}{m_\phi^2}}$$

Srednicki 11.2. Consider *Compton Scattering*, in which a massless photon is scattered by an electron, initially at rest (this is the FT frame). In problem 59.1, we will compute $|\tau|^2$ for this process (summed over the possible spin states of the scattered photon and electron, and averaged over the possible spin states of the initial photon and electron), with the result:

$$|\boldsymbol{\tau}|^2 = \mathbf{32\pi^2\alpha^2\left[\frac{m^4+m^2(3s+u)-su}{(m^2-s)^2}+\frac{m^4+m^2(3u+s)-su}{(m^2-u)^2}+\frac{2m^2(s+u+2m^2)}{(m^2-s)(m^2-u)}\right]+O(\alpha^4)}$$

where $\boldsymbol{\alpha} = \mathbf{1/137.036}$ is the fine-structure constant.

(a) Express the Mandelstam variables s and u in terms of the initial and final photon energies $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$.

Using equation 11.5,

$$s = -(k_1+k_2)^2 = -k_1^2 - 2k_1k_2 - k_2^2$$

k_1 is the photon so $k_1^2 = -m_1^2 = 0$. k_2 is the electron, so $k_2^2 = -m^2$, where we'll use m to represent the mass of the electron. Thus,

$$s = m^2 - 2k_1k_2$$

Now $k_1 = \begin{pmatrix} \omega \\ \vec{\omega} \end{pmatrix}$ and $k_2 = \begin{pmatrix} m \\ 0 \end{pmatrix}$. Hence,

$$\boxed{s = m^2 + 2m\omega}$$

Similarly, $u = (-k_2 - k_1')^2$, so

$$\boxed{u = m^2 - 2\omega' m}$$

(b) Express the scattering angle $\boldsymbol{\theta_{FT}}$ between the initial and final photon three-momenta in terms of $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$.

By conservation of four-momentum:

$$(k_2')^2 = (k_1 + k_2 - k_1')^2$$

which gives:

$$-m^2 = k_1^2 + k_2^2 + k_1'^2 + 2k_1 \cdot k_2 - 2k_2 \cdot k_1' - 2k_1 \cdot k_1'$$

Hence:

$$(\omega - \omega')m = (\omega\omega' - \vec{\omega} \cdot \vec{\omega'})$$

$$1 - cos\theta = m\frac{\omega - \omega'}{\omega\omega'}$$

$$\boxed{cos\ \theta = 1 - m\left(\frac{1}{\omega'} - \frac{1}{\omega}\right)}$$

(c) Express the differential scattering cross-section (in the FT frame) in terms of $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$. Show that your result is equivalent to the *Klein-Nishina* formula

$$\mathbf{\frac{d\sigma}{d\Omega_{FT}} = \frac{\boldsymbol{\alpha}^2}{2m^2}\frac{\boldsymbol{\omega}'^2}{\boldsymbol{\omega}^2}\left[\frac{\boldsymbol{\omega}}{\boldsymbol{\omega}'} + \frac{\boldsymbol{\omega}'}{\boldsymbol{\omega}} - \sin^2\boldsymbol{\theta}_{FT}\right]}$$

Using the chain rule:

$$\frac{d\sigma}{d\Omega_{FT}} = \frac{d\sigma}{dt}\frac{dt}{d\Omega_{FT}} \tag{11.2.1}$$

We'll deal with each term separately, starting with the first term. Using the result from part (a), we rewrite the matrix element in terms of ω and ω'. This is a horrible calculation, but the result is:

$$|\tau|^2 = 32\pi^2\alpha^2\left[m^2\left(\frac{1}{\omega^2} + \frac{1}{\omega'^2} - \frac{2}{\omega\omega'}\right) + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + \frac{\omega'}{\omega} + \frac{\omega}{\omega'}\right]$$

Now use the result from part (b). The first and second terms are rewritten:

$$|\tau|^2 = 32\pi^2\alpha^2\left[1 - 2\cos\theta + cos^2\theta + 2\cos\theta - 2 + \frac{\omega'}{\omega} + \frac{\omega}{\omega'}\right]$$

Rewriting:

$$|\tau|^2 = 32\pi^2\alpha^2\left[-1 + cos^2\theta + \frac{\omega'}{\omega} + \frac{\omega}{\omega'}\right]$$

Using the Pythagorean Identity:

$$|\tau|^2 = 32\pi^2\alpha^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta\right] \tag{11.2.2}$$

Now we use equation 11.34:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{k}_1|^2}|\tau|^2$$

This is why equation 11.9 is useful – we can rewrite this as:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi m^2 |\vec{k_1}|_{CM}} |\tau|^2$$

Now $|\vec{k_1}|_{CM}$ refers to the photon, so:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi m^2 \omega^2} |\tau|^2$$

Plugging in equation (11.2.2), we have:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi m^2 \omega^2} 32\pi^2 \alpha^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta \right]$$

which is:

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{2m^2\omega^2} \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta \right] \tag{11.2.3}$$

The other part of equation (11.2.1) is obtained starting with equation 11.4:

$$t = m_1^2 + m_1'^2 - 2E_1E_1' + 2|k_1||k_1'|cos\theta$$

Taking the differential:

$$dt = 2\omega\omega' dcos\theta - (1 - cos\theta)2\omega d\omega'$$

Note that holding s constant entails holding ω constant (according to equation 11.9). Next we'll take the differential of the result from part (b), with the result that $d\omega' = \frac{(\omega')^2}{m} dcos\theta$. Hence,

$$dt = 2\omega\omega' dcos\theta - (1 - cos\theta)2\omega \frac{(\omega')^2}{m} dcos\theta$$

Now use the result of part (b) to substitute for $(1 - cos\theta)$:

$$dt = 2\omega\omega' dcos\theta - m\left(\frac{1}{\omega'} - \frac{1}{\omega}\right) 2\omega \frac{(\omega')^2}{m} dcos\theta$$

Now we simplify:

$$dt = 2\omega'^2 dcos\theta$$

which, according to the convention used between equations 11.32 and 11.33, gives:

$$dt = \frac{\omega'^2}{\pi} d\Omega$$

It may seem as though we're off by a negative sign, but in fact we're not. Equation 11.4 has many square roots in it, which essentially gives us the power to assign the signs by hand. Cross-sections must always be positive, so we choose to ignore the negative sign generated in evaluating $dcos\theta$. Hence, the result is that:

$$\frac{dt}{d\Omega} = \frac{\omega'^2}{\pi} \tag{11.2.4}$$

Using (11.2.3) and (11.2.4) in (11.2.1), we find:

$$\frac{d\sigma}{d\Omega} = \frac{\omega'^2}{\pi}\frac{\pi\alpha^2}{2m^2\omega^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta\right]$$

which is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\omega'^2}{2m^2\omega^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta\right]$$

which is the Klein-Nishina formula.

Srednicki 11.3. Consider the process of *muon decay*, $\boldsymbol{\mu_- \to e^- \bar{\nu}_e \nu_\mu}$. In section 88, we will compute $|\boldsymbol{\tau}|^2$ for this process (summed over the possible spin states of the decay products, and averaged over the possible spin states of the initital muon), with the result:

$$|\boldsymbol{\tau}|^2 = \mathbf{64G_F^2(k_1 \cdot k_2')(k_1' \cdot k_3')}$$

where $\mathbf{G_F}$ is the *Fermi Constant*, $\mathbf{k_1}$ is the four-momentum of the muon, and $\mathbf{k'_{1,2,3}}$ are the four-momenta of the $\bar{\nu}_e$, $\boldsymbol{\nu_\mu}$, and $\boldsymbol{e^-}$, respectively. In the rest frame of the muon, its decay rate is therefore:

$$\mathbf{\Gamma = \frac{32G_F^2}{m}\int (k_1 \cdot k_2')(k_1' \cdot k_3')dLIPS_3(k_1)}$$

where $\mathbf{k_1 = (m, 0)}$, and $\boldsymbol{m}$ is the muon mass. The neutinos are taken to be massless, and the electron mass is 200 times less than the muon mass, so we can take the electron to be massless as well. To evaluate Γ, we perform the following analysis.

(a) Show that:

$$\mathbf{\Gamma = \frac{32G_F^2}{m}\int \widetilde{dk}_3' k_{1\mu}k'_{3\nu}\int k_2'^{\mu}k_1'^{\nu}dLIPS_2(k_1 - k_3')}$$

We start with the given:

$$\Gamma = \frac{32G_F^2}{m}\int (k_1 \cdot k_2')(k_1' \cdot k_3')dLIPS_3(k_1)$$

Then we simply use equation 11.23:

$$dLIPS_3(k_1) = (2\pi)^4\delta^4(k_1 - k_1' - k_2' - k_3')\widetilde{dk}_1'\widetilde{dk}_2'\widetilde{dk}_3'$$

$$dLIPS_2(k_1 - k_3) = (2\pi)^4\delta^4(k_1 - k_1' - k_2' - k_3')\widetilde{dk}_1'\widetilde{dk}_2'$$

From which it follows that we can rewrite Γ as:

$$\Gamma = \frac{32G_F^2}{m}\int (k_1 \cdot k_2')(k_1' \cdot k_3')dLIPS_2(k_1 - k_3')\widetilde{dk}_3'$$

or, equivalently:

$$\Gamma = \frac{32G_F^2}{m}\int \widetilde{dk}_3' k_{1\mu} k'_{3\nu} \int k_2'^{\mu} k_1'^{\nu} dLIPS_2(k_1 - k_3')$$

(b) Use Lorentz Invariance to argue that, for $\mathbf{m_{1'} = m_{2'} = 0}$,

$$\int \mathbf{k_1'^{\mu} k_2'^{\nu} dLIPS_2(k) = Ak^2 g^{\mu\nu} + Bk^{\mu}k^{\nu}}$$

The left hand side of this equation is a tensor with two indices. It can depend on only one four vector: k^{μ}. It cannot depend on k_1^{μ} or k_2^{μ} – for any given μ, ν, those terms will simply contribute a (frame-dependent) constant. Hence,

$$\int k_1'^{\mu} k_2'^{\nu} dLIPS_2(k) = Ak^2 g^{\mu\nu} + Bk^{\mu}k^{\nu}$$

By dimensional analysis, A and B must be dimensionless. Everything except the index-holders is required to be Lorentz-invariant in any case, and the only Lorentz-invariant scalar that's possible is $k^2 = -m^2$, which is not dimensionless. Hence, A and B are just numbers.

(c) Show that, for $\mathbf{m_{1'} = m_{2'} = 0}$,

$$\int \mathbf{d\mathrm{LIPS}_2(k) = \frac{1}{8\pi}}$$

Using equation 11.30:

$$d\mathrm{LIPS}_2(k) = \frac{|\vec{k}_1'|}{16\pi^2\sqrt{s}} d\Omega_{CM}$$

Of course, $\sqrt{s} = m_\mu$ for decay. Further, $|k_1'| = \frac{m_\mu}{2}$ for decay into two massless particles (in the CM frame). Then,

$$d\mathrm{LIPS}_2(k) = \frac{m_\mu}{32\pi^2 m_\mu} d\Omega_{CM}$$

$$\implies \int d\mathrm{LIPS}_2(k) = \frac{1}{32\pi^2}\int d\Omega_{CM} = \frac{1}{8\pi}$$

(d) By contracting both sides of equation 11.55 with $\mathbf{g_{\mu\nu}}$ and with $\mathbf{k_\mu k_\nu}$, and using equation 11.56, evaluate A and B.

Contracting with $g_{\mu\nu}$ first, we have:

$$\int (k_1' \cdot k_2') d\mathrm{LIPS}_2(k) = 4Ak^2 + Bk^2$$

Now note that:

$$k^2 = (k_1' + k_2')^2 = k_1'^2 + k_2'^2 = 2k_1' \cdot k_2' \tag{11.3.1}$$

Hence,

$$\frac{1}{2}k^2 \int d\mathrm{LIPS}_2(k) = 4Ak^2 + Bk^2$$

Using the result of part (c), and canceling the factors of k^2 (this is a Lorentz-invariant constant, so it must have an inverse).

$$\frac{1}{16\pi} = 4A + B \tag{11.3.2}$$

Now let's go back to equation 11.55 and contract with $k_\mu k_\nu$:

$$\int (k_1' \cdot k)(k_2' \cdot k) d\text{LIPS}_2(k) = Ak^4 + Bk^4$$

Using equation (11.3.1):

$$\int \frac{1}{2}k^2 \frac{1}{2}k^2 \int d\text{LIPS}_2(k) = Ak^4 + Bk^4$$

Canceling the k^4 and using the result of part (c)

$$\frac{1}{32\pi} = A + B \tag{11.3.3}$$

We now have a system of two equations and two unknows (equations (11.3.2) and (11.3.3)). Doing the algebra, we find

$$\boxed{\begin{array}{rcl} A & = & \frac{1}{96\pi} \\ \\ B & = & \frac{1}{48\pi} \end{array}}$$

(e) Use the results of parts (b) and (d) in equation 11.54. Set $\mathbf{k_1} = (\mathbf{m}, \tilde{\mathbf{0}})$ and compute $\mathbf{d\Gamma/dE_e}$; here $\mathbf{E_e} = \mathbf{E_3'}$ is the energy of the electron. Note that the maximum value of $\mathbf{E_e}$ is reached when the electron is emitted in one direction, and the two neutrinos in the opposite direction; what is this maximum value?

The result of part (a) is:

$$\Gamma = \frac{32G_F^2}{m} \int \widetilde{dk}_3' k_{1\mu} k_{3\nu}' \int k_2'^\mu k_1'^\nu dLIPS_2(k_1 - k_3')$$

Inserting the result of part (b):

$$\Gamma = \frac{32G_F^2}{m} \int \widetilde{dk}_3' k_{1\mu} k_{3\nu}' \left(A(k_1 - k_3')^2 g^{\mu\nu} + B(k_1 - k_3')^\mu (k_1 - k_3')^\nu\right)$$

Inserting the result of part (d):

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk}_3' k_{1\mu} k_{3\nu}' \left((k_1 - k_3')^2 g^{\mu\nu} + 2(k_1 - k_3')^\mu (k_1 - k_3')^\nu\right)$$

Next we simplify:

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk}_3' k_{1\mu} k_{3\nu}' \left((k_1^2 + k_3'^2 - 2k_1 \cdot k_3')g^{\mu\nu} + 2k_1^\mu k_1^\nu - 2k_1^\mu k_3'^\nu - 2k_3'^\mu k_1^\nu + 2k_3'^\mu k_3'^\nu\right)$$

$k_1^2 = -m^2$ and $k_3'^2 = 0$. Further, we decided to work in the rest frame of the muon, so $k_1 \cdot k_3' = -mE_e$. So:

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk_3'} k_{1\mu} k'_{3\nu} \left((-m^2 + 2mE_e) g^{\mu\nu} + 2k_1^\mu k_1^\nu - 2k_1^\mu k_3'^\nu - 2k_3'^\mu k_1^\nu + 2k_3'^\mu k_3'^\nu \right)$$

Now we'll distribute the four-vectors in front of the term:

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk_3'} \left((-m^2 + 2mE_e) k_{1\mu} k'_{3\nu} g^{\mu\nu} + 2k_{1\mu} k'_{3\nu} k_1^\mu k_1^\nu - 2k_{1\mu} k'_{3\nu} k_1^\mu k_3'^\nu - 2k_{1\mu} k'_{3\nu} k_3'^\mu k_1^\nu + 2k_{1\mu} k'_{3\nu} k_3'^\mu k_3'^\nu \right)$$

which gives:

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk_3'} \left((-m^2 + 2mE_e)(k_1 \cdot k_3') + 2(k_1^2)(k_1 \cdot k_3') - 2(k_1^2)(k_3'^2) - 2(k_1 \cdot k_3')^2 + 2(k_1 \cdot k_3')(k_3'^2) \right)$$

Now recall that $k_1 \cdot k_3' = -mE_e$ and $k_3^2 = 0$ and $k_1^2 = -m^2$. Then:

$$\Gamma = \frac{G_F^2}{3\pi m} \int \widetilde{dk_3'} \left((m^3 E_e - 2m^2 E_e^2) + 2m^3 E_e - 2m^2 E_e^2 \right)$$

which is:

$$\Gamma = \frac{mG_F^2}{\pi} \int \widetilde{dk_3'} \left(E_e m - \frac{4}{3} E_e^2 \right)$$

Now we expand the differential:

$$\Gamma = \frac{mG_F^2}{\pi} \int \frac{d^3 k_3'}{(2\pi)^3 2E_e} \left(E_e m - \frac{4}{3} E_e^2 \right)$$

and switch to polar coordinates:

$$\Gamma = \frac{mG_F^2}{\pi} \int \frac{E_e^2 dE_e d\Omega}{(2\pi)^3 2E_e} \left(E_e m - \frac{4}{3} E_e^2 \right)$$

The angular integrals are trivial, since the electron emission is isotropic:

$$\Gamma = \frac{mG_F^2}{\pi (2\pi)^2} \int dE_e \left(E_e^2 m - \frac{4}{3} E_e^3 \right)$$

Now let's write this as a differential:

$$\boxed{\frac{d\Gamma}{dE_e} = \frac{mG_F^2}{4\pi^3} \left(E_e^2 m - \frac{4}{3} E_e^3 \right)}$$

This is the distribution of the electron energy. We want to calculate the maximum electron energy, so we set the derivative of this equal to zero:

$$0 = \frac{mG_F^2}{(2\pi)^2} \left(2E_e m - 4E_e^2 \right)$$

and solve:

$$0 = (2m - 4E_e)$$

and so:

$$\boxed{E_{e,max} = \frac{m}{2}}$$

Just as the problem statement says, conservation of four-momentum in the muon's rest frame tells us that if the electron gets the maximum amount of energy (half), then the other two particles will both have to move in the opposite direction for four-momentum to be conserved.

(f) Perform the integral over $\mathbf{E_e}$ to obtain the muon decay rate $\mathbf{\Gamma}$.

This is just an integral; the only thing to note is that the limits of integration are from 0 to $\frac{m}{2}$. The result is:

$$\Gamma = \frac{mG_F^2}{4\pi^3}\int_0^{m/2} dE_e\left(E_e^2 m - \frac{4}{3}E_e^3\right)$$

$$\implies \boxed{\Gamma = \frac{m^5 G_F^2}{192\pi^3}}$$

(g) The measured lifetime of the muon is $\mathbf{2.197 \times 10^{-6}}$ s. The muon mass is 105.66 MeV. Determine the value of $\mathbf{G_F}$ in $\mathbf{GeV^{-2}}$. Your answer is too low by about 0.2%, due to loop corrections to the decay rate.

$\tau = \frac{1}{\Gamma}$, so $\Gamma = 455,166.135\ \mathrm{s}^{-1}$. Multiplying by Planck's constant (to change units), we find that $\Gamma = 2.9959 \times 10^{-19}$ GeV.

Now from the result of part (f):

$$G_F = \sqrt{\frac{192\pi^3\Gamma}{m^5}}$$

Plugging in numbers:

$$G_F = \sqrt{\frac{192\pi^3(2.9959 \times 10^{-19}\ \mathrm{GeV})}{(.10566^5\ \mathrm{GeV}^5)}}$$

The result is that:

$$\boxed{G_F = 1.164 \times 10^{-5}\ \mathrm{GeV}^{-2}}$$

The accepted value is $1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$, which is about .2% higher as advertised.

(h) Define the *energy spectrum* of the electron $\mathbf{P(E_e) = \Gamma^{-1}d\Gamma/dE_e}$. Note that $\mathbf{P(E_e)dE_e}$ is the probability for the electron to be emitted with energy between $\mathbf{E_e}$ and $\mathbf{E_e + dE_e}$. Draw a graph of $\mathbf{P(E_e)}$ versus $\mathbf{E_e/m_\mu}$.

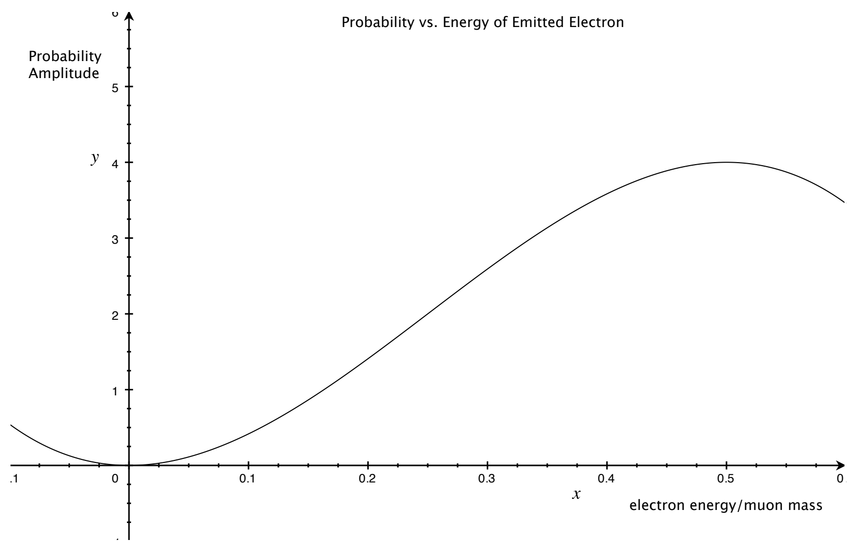
$P(E_e)$ is given by:

$$P(E_e) = \frac{48}{m}\left(\frac{E_e^2}{m^2} - \frac{4}{3}\frac{E_e^3}{m^3}\right)$$

Integrating this from $E_e = 0$ to $E_e = \frac{m}{2}$ gives one, as we would expect. Now we switch variables, from $E_e \to x = \frac{E_e}{m}$. Note that if this new variable is to represent a probability, then this integration measure must also switch, $dE_e \to m dx$. Otherwise, $\int P(x)dx \neq 1$, which is a problem. Then,

$$P(x) = 48\left(x^2 - \frac{4}{3}x^3\right)$$

This problem's notation is a little bit confusing: we are not actually plotting $P(E_e)$, but rather $P(E_e/m)$. Nonetheless, the physical significance is still the same, so this is an appropriate answer to the question. Here is the graph:



By the way, don't be alarmed that the y-axis is greater than 1! The probability is defined to be the integral of the amplitude between 0 and m/2, and the maximum possible such integral is one. In other words, the probability density can be as high as we like.

We conclude that the electron is most likely to receive the maximum amount (half) of the energy. This is an interesting result; we might have naively assumed that the most likely outcome was all three particles receiving the same amount of energy, and going in three uncorrelated directions.

Srednicki 11.4. Consider a theory of three real scalar fields A, B, and C with:

$$\boldsymbol{\mathcal{L} = -\frac{1}{2}\partial^\mu A\partial_\mu A - \frac{1}{2}m_A^2 A^2 - \frac{1}{2}\partial^\mu B\partial_\mu B - \frac{1}{2}m_B^2 B^2 - \frac{1}{2}\partial^\mu C\partial_\mu C - \frac{1}{2}m_C^2 C^2 + gABC}$$

Write down the tree-level scattering amplitude (given by the sum of the contributing tree diagrams) for each of the following processes:

$$\mathbf{AA \to AA}$$

$$\mathbf{AA \to AB}$$

$$\mathbf{AA \to BB}$$

$$\mathbf{AA} \to \mathbf{BC}$$

$$\mathbf{AB} \to \mathbf{AB}$$

$$\mathbf{AB} \to \mathbf{AC}$$

For two-body scattering, the only tree-level diagrams are those shown in figure 10.2 in the text: s-channel, t-channel, and u-channel, all with two vertices. We can see by inspecting the Lagrangian that this theory has only one vertex. $\mathcal{L}_{int} = gABC$, so the vertex must join an A, a B, and a C propagator, with a vertex factor of ig (just rub out the fields, there are no derivatives and no symmetry factors).

Now we try to draw these diagrams: it quickly becomes obvious that there are no diagrams for the first, second, fourth, or sixth processes. Hence,

$$\boxed{\tau_{AA\to AA} = 0}$$

$$\boxed{\tau_{AA\to AB} = 0}$$

$$\boxed{\tau_{AA\to BC} = 0}$$

$$\boxed{\tau_{AB\to AC} = 0}$$

The second process has two diagrams, the t-channel and the u-channel. The value of each diagram (iτ) is given by the vertex factor (ig) as well as the internal propagator. The internal propagator is always a C scalar, but the momentum is different: in the t-channel, the momentum is $k_1 - k_1'$, while in the u-channel, it is $k_1 - k_2'$. This is a good time to use the Mandelstam variables. Using the Feynman rules from chapter 10, the internal propagators contribute $\frac{-i}{m_C^2 - t}$ and $\frac{-i}{m_C^2 - u}$, respectively. Putting all this together, we have:

$$i\tau = ig^2 \frac{1}{m_C^2 - t} + ig^2 \frac{1}{m_C^2 - u}$$

Note that we do not have to sum over "multiple copies" of the same diagram. For example, the t-channel diagram cannot have the A scalars flipped, since that turns it into a u-channel diagram. Both the A scalars and B scalars could be flipped, but then we've recovered a rotation of the original diagram.

In fact, this is a general rule for four-particle tree diagrams such as these: in calculating Z(J), there are 24 ways to pair derivatives to sources, 8 for each diagram. The factor of 8 is cancelled by the symmetry factors of the diagrams with sources, so each diagram is counted exactly once.

Hence,

$$\boxed{\tau_{AA\to BB} = g^2 \left(\frac{1}{m_C^2 - t} + \frac{1}{m_C^2 - u} \right)}$$

$$\boxed{\tau_{AB\to AB} = g^2 \left(\frac{1}{m_C^2 - s} + \frac{1}{m_C^2 - u} \right)}$$