Srednicki Chapter 10

QFT Problems & Solutions

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Srednicki 10.1. Use equation 9.41 to rederive equation 10.9.

Equation 9.41 is:

$$\langle 0|T\phi(x_n)\dots\phi(x_1)|0\rangle = \frac{\langle\emptyset|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}{\langle\emptyset|Te^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}$$

In our case, we have four terms, so:

$$\langle 0|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0\rangle = \frac{\langle\emptyset|T\phi_I(x_4)\phi_I(x_3)\phi_I(x_2)\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}{\langle\emptyset|Te^{-i\int d^4x\mathcal{H}_I(x)}|\emptyset\rangle}$$

Next, we need to figure out what H_I is. Equation 9.4 gives us the Hamiltonian, of which the first, second, and fourth of which comprise H_0 and the counterm Lagrangian. Note that equation 10.9 only goes through second order, and the counterterms are already O(g^2) – so there's no way to build a connected diagram involving a counterterm at the order to which we are working.

All of which is just to say that $\mathcal{H}_I = -Y\phi - \frac{1}{6}Z_g g\phi^3$. But we can make this even easier: since we want to ignore tadpoles, we don't need this first term. Hence, $\mathcal{H}_I = -\frac{1}{6}Z_g g\phi^3$. But don't forget our result in part (d) of problem 9.5, which shows that our ϕ terms should really be ϕ_I terms (that's also why we're allowed to use the Heisenberg-picture Hamiltonian in the Interaction-picture expression). Hence, the right side of our expression becomes:

$$\Longrightarrow \frac{\langle\emptyset|T\phi_I(x_4)\phi_I(x_3)\phi_I(x_2)\phi_I(x_1)\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{i}{6}Z_g g\int d^4x\phi_I^3(x)\right)^N|\emptyset\rangle}{\langle\emptyset|T\sum_{M=0}^{\infty}\frac{1}{M!}\left(\frac{i}{6}Z_g g\int d^4x\phi^3(x)\right)^M|\emptyset\rangle}$$

Let's notice again that equation 10.9 only goes through second-order. Moreover, the first-order terms involve an odd number of fields, and therefore vanish by Wick's Theorem (same for third-order terms). So, we can expand, leaving just the zero-order and second-order terms:

$$\Longrightarrow \frac{\langle\emptyset|T\phi_I(x_4)\phi_I(x_3)\phi_I(x_2)\phi_I(x_1)\left(1+\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y\phi_I^3(x)\phi_I^3(y)+O(g^4)\right)|\emptyset\rangle}{\langle\emptyset|T\left(1+\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y\phi_I^3(x)\phi_I^3(y)+O(g^4)\right)|\emptyset\rangle}$$

For simplicity, I'm going to supress the factors of I in the subscript. But it is important to remember that these are still free fields.

$$\Longrightarrow \frac{\langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\left(1+\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y\phi^3(x)\phi^3(y)+O(g^4)\right)|\emptyset\rangle}{\langle\emptyset|T\left(1+\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y\phi^3(x)\phi^3(y)+O(g^4)\right)|\emptyset\rangle}$$

Now we're ready to expand the denominator. Recall that $(1+x^2)^{-1} = 1 - x^2 + \ldots$. We are only working to O(g^2), so we simply move the denominator to the numerator and swap the sign. Hence,

$$\Longrightarrow \left[\langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\left(1+\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y \phi^3(x)\phi^3(y) + O(g^4)\right)|\emptyset\rangle\right]\times$$

$$\left[\langle\emptyset|T\left(1-\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y \phi^3(x)\phi^3(y) + O(g^4)\right)|\emptyset\rangle\right]$$

Multiplying these, we get one term of order zero and two terms of order two. The zero-order term doesn't have any vertices, so it cannot possibly be a connected diagram. Hence, we ignore that term, and keep only the O(g^2) terms (all other terms will be of order g^4 or higher).

$$\Longrightarrow \langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\,(1)\,|\emptyset\rangle\times\langle\emptyset|T\frac{-(i^2)}{6^2 2!}Z_g^2 g^2\int d^4x d^4y \phi^3(x)\phi^3(y)|\emptyset\rangle$$

$$+\langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\left(\frac{i^2}{6^2 2!}Z_g^2 g^2\int d^4x d^4y \phi^3(x)\phi^3(y)\right)|\emptyset\rangle\times\langle\emptyset|T\,(1)\,|\emptyset\rangle$$

This is a good opportunity to make some simplifications. Get rid of the i terms and notice that the last term in the above expression is one. Additionally, let's choose different variables for the interaction terms so that we can tell them apart. Let's define $y_1 = y_2 = y_3 = y$, and the same for x – but to avoid confusion, let's reassign $x \to z = z_1 = z_2 = z_3$. Then,

$$\Longrightarrow \frac{Z_g^2 g^2}{6^2 2!}\int d^4y d^4z \langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|\emptyset\rangle\langle\emptyset|T\phi(y_1)\phi(y_2)\phi(y_3)\phi(z_1)\phi(z_2)\phi(z_3)|\emptyset\rangle$$

$$+\frac{-Z_g^2 g^2}{6^2 2!}\int d^4y d^4z \langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\phi(y_1)\phi(y_2)\phi(y_3)\phi(z_1)\phi(z_2)\phi(z_3)|\emptyset\rangle$$

Now we're ready to use Wick's Theorem, equation 8.17. Remember that we can use Wick's Theorem because our terms marked ϕ are actually ϕ_I, and we proved in problem 9.5(a) that the ϕ_Is are free fields. The problem is that there are many possible pairings! We can simplify a lot by dividing the second term in two parts: one part with all the x terms paired with each other, and one part with at least one x paired with a y or a z. We see then that this first part cancels with the first term. Hence,

$$\Longrightarrow \frac{-Z_g^2 g^2}{6^2 2!}\int d^4y d^4z \langle\emptyset|T\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)\phi(y_1)\phi(y_2)\phi(y_3)\phi(z_1)\phi(z_2)\phi(z_3)|\emptyset\rangle^*$$

where the * is to remind us that the x's are not allowed to be paired exclusively with each other. Now let's apply Wick's Theorem:

$$\Longrightarrow \frac{-Z_g^2 g^2}{6^2 2!}\frac{1}{i^5}\int d^4y d^4z \sum_{pairings*}\Delta(x_1-y_1)\Delta(x_2-x_3)\Delta(x_4-y_2)\Delta(y_3-z_1)\Delta(z_2-z_3)$$

To make this look a little more like equaton 10.9, let's recall that $Z_g = 1 + O(g^2)$. We don't have any zero-order terms to which we could add the nontrivial part of Z_g, so Z_g will equal one to the order in which we are working. So,

$$\Longrightarrow (ig)^2\left(\frac{1}{i}\right)^5\frac{1}{6^2 2!}\int d^4y d^4z \sum_{pairings*}\Delta(x_1-y_1)\Delta(x_2-x_3)\Delta(x_4-y_2)\Delta(y_3-z_1)\Delta(z_2-z_3)$$

Next, remember that we want only connected diagrams. In this case, that means that *none* of the xs should be paired with each other: they should all be paired with an interaction term in some way. Further, none of the y (z) terms should be paired with another y (z) term, since that will yield a disconnected diagram, with a loop on one vertex and three external lines on the other vertex. This implies that all diagrams will have to have a vertex of $\Delta(y_i - z_j)$, with the other ys and zs paired to xs.

So, which pairings are allowed? We're allowed $\Delta(y_1 - z_1)\Delta(x_1 - y_2)\Delta(x_2 - y_3)\Delta(x_3 - z_2)\Delta(x_4 - z_3)$. We can switch the three ys and the three zs around, so this term shows up $3!\ 3! = 36$ times. Additionally, we can swap $y_i \longleftrightarrow z_i$; this will not be significant after integration. So, we get 72 copies of this term: $\Delta(y-z)\Delta(x_1-y)\Delta(x_2-y)\Delta(x_3-z)\Delta(x_4-z)$.

The only thing we haven't accounted for is my arbitrary decision to pair x_1 and x_2 with y while pairing x_3 and x_4 with z. These are dummy variables, so it's acceptable to arbitrarily pair x_1 with y (because I can just redefine the dummy variable if I want to pair it with z). But, I have to allow for the swap $x_2 \longleftrightarrow x_3$ or $x_2 \longleftrightarrow x_4$ (because I can't redefine the dummy variable in some places but not others). These terms are not identical to the first term, and I will get 72 copies of each. Hence,

$$\implies (ig)^2 \left(\frac{1}{i}\right)^5 \int d^4y d^4z \Delta(y-z) \times [\Delta(x_1-y)\Delta(x_2-y)\Delta(x_3-z)\Delta(x_4-z)$$

$$+\Delta(x_1-y)\Delta(x_3-y)\Delta(x_2-z)\Delta(x_4-z) + \Delta(x_1-y)\Delta(x_4-y)\Delta(x_3-z)\Delta(x_2-z)]$$

which is equation 10.9.

Srednicki 10.2. Write down the Feynman rules for the complex scalar field of problem 9.3.

We've had several problems involving this theory, so let's summarize what we know before moving on.

The relationship between a, b, $a^\dagger$, $b^\dagger$, ϕ, $\phi^\dagger$, and the arrows:

- incoming a particles ($a^\dagger$) have $\phi^\dagger$ expansions, and arrows that point toward the vertex.
- incoming b particles ($b^\dagger$) have ϕ expansions, and arrows that point away from vertex.
- outgoing a particles (a) have ϕ expansions, and arrows that point away from the vertex.
- outgoing b particles (b) have $\phi^\dagger$ expansions, and arrows that point toward the vertex.

How do we know all this? We worked out the expansions in problem 5.1 and 3.5. In problem 8.7, we took as our source term $J\phi^\dagger + J^\dagger\phi$, and set arrows toward the source for $J^\dagger$ and away from the source for J. After differentiating with respect to the source, we're left with arrows toward the (missing) source for ϕ and away from the (missing) source for $\phi^\dagger$. Since the source is at the opposite end than the vertex, we therefore achieve the above arrow conventions.

We should also recall our results involving the vertex, from problem 9.3:

- The vertex joins two ϕ particles and two $\phi^\dagger$ particles. From the above, we see that we must have two arrows pointing toward the vertex, and two arrows pointing away from the vertex.

- The vertex factor is $-iZ_\lambda\lambda$.

Notice that I no longer include the integral in the vertex factor (this is a change from my solution to problem 9.3 – see the slides for further discussion). This is a subtle and perhaps unintentional change in Srednicki's convention – previously, the entire expansion of Z(J) (integrals and all) had to be represented with the diagram – and so the vertex factor had to include the integral (see for example eq. 9.11). But in this section, and for the rest of the book, the integrals are included separately (in the LSZ formula, for example), it is not necessary to represent them in the vertex factors. That's why Srednicki specifes 12 lines below eq. 9.11 that the vertex factor includes the integral, but then seems to change his mind in Feynman Rule #6 on page 77.

Next we have this business involving the two types of arrows: the charge arrows discussed above, and the momentum arrows that are typically assigned toward the vertex for incoming particles, and away from the vertex for outgoing particles. Notice that the a particles already follow this convention, so we can just use the charge arrows for the momentum arrows. b particles follow the opposite of this convention, but all is well if we assign a negative momentum along the direction of the charge arrow.

Hence, let us write down the Feynman Rules:

1. Draw lines (called external lines) for each incoming and each outgoing particle. The lines must have arrows on them:

   - Arrows toward the vertex for incoming a and outgoing b particles.
   - Arrows away from the vertex for incoming b and outgoing a particles.

2. Leave one end of each external line free, and attach the other end to a vertex at which exactly four lines meet. This vertex must have two arrows pointing toward the vertex and two arrows pointing away. Include extra internal lines in order to do this, assigning whichever arrow is needed to complete the vertex. In this way, draw all possible diagrams that are topologically inequivalent.

3. Assign each line its own four-momentum. The four momentum of an external line should be the four momentum of the corresponding particle. Assign a particles a positive four-momentum, and b particles a negative four-momentum.

4. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree-diagram, this fixes the momenta on all the internal lines.

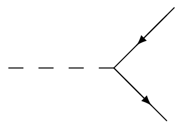
5. The value of a diagram consists of the following factors:

   - for each external line, 1;
   - for each internal line with momentum k, $-i/(k^2 + m^2 - i\epsilon)$
   - for each vertex, -i$Z_\lambda\lambda$

6. A diagram with L closed loops will have L internal momenta that are not fixed by rule no. 4. Integrate over each of these momenta ℓ_i with measure $d^4\ell_i/(2\pi)^4$.
7. A loop diagram may have some left-over symmetry factors if there are exchanges of internal propagators and vertices that leave the diagram unchanged; in this case, divide the value of the diagram by the symmetry factor associated with exchanges of internal popagators and vertices.
8. Include diagrams with the counterterm vertex that connects two propagators, each with the same four-momentum k. The value of this vertex is $-i(Ak^2 + Bm^2)$, where $A = Z_\phi - 1$ and $B = Z_m - 1$, and each is $O(g^2)$.
9. The value of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

Srednicki 10.3. Consider a complex scalar field ϕ that interacts with a real scalar field χ via $\mathcal{L}_1 = g\chi\phi^\dagger\phi$. Use a solid line for the ϕ propogator and a dashed line for the χ propagator. Draw the vertex (remember the arrows!), and find the associated vertex factor.

Drawing the vertex is the easy part. Recall that we defined in problem 9.3 that ϕ propagators point away from the vertex, and $\phi^\dagger$ vertices point toward the vertex. Thus, the vertex is:



Now for the vertex factor. We'll do this in analogy to the ϕ^3 case in the text. In the case of ϕ^3, we had a Lagrangian of $\frac{1}{6}Z_g g\phi^3$ and a vertex factor of $iZ_g g$.

In our case, the Lagrangian is $g\chi\phi^\dagger\phi$, so we might expect the vertex factor to be $6ig$. But looking closely over chapter 9, we see that Srednicki treated the numerical factors separately from the vertex factors. This turned out to be with good reason, because there are 3! identical ways in which the vertex legs could pair with the propagators, neatly canceling the factor of 6. In our case, the numerical factor is one, but there is also only one identical way in which the vertex legs can pair with the propagators. So, the numerical factors again neatly cancel, and the vertex factor is in fact ig.

Srednicki 10.4. Consider a real scalar field with $\mathcal{L}_1 = \frac{1}{2}g\phi\partial^\mu\phi\partial_\mu\phi$. Find the associated vertex factor.

The interaction Lagrangian is:

$$\mathcal{L}_1 = \frac{1}{2}g\phi\partial^\mu\phi\partial_\mu\phi$$

Now we'll take the Fourier Transforms. Note that we assume that the ks are positive, meaning that the (momentum) arrows are all coming toward the vertex. If any point away from the vertex, the corresponding k in the vertex factor will be made negative.

$$\mathcal{L}_1 = \frac{1}{2}g\int d^4k_1 d^4k_2 d^4k_3 e^{ik_1x_1}\widetilde{\phi}(k_1)\partial_2^\mu e^{ik_2x_2}\widetilde{\phi}(k_2)\partial_{\mu,3}e^{ik_3x_3}\widetilde{\phi}(k_3)$$

Next we'll determine vertex factors, using the prescription presented in the slides:

$$\text{V.F.} = i\frac{\delta}{\delta\widetilde{\phi}(k_1)}\frac{\delta}{\delta\widetilde{\phi}(k_2)}\frac{\delta}{\delta\widetilde{\phi}(k_3)}\frac{1}{2}g\int d^4k_1d^4k_2d^4k_3e^{ik_1x_1}\widetilde{\phi}(k_1)\partial_2^{\mu}e^{ik_2x_2}\widetilde{\phi}(k_2)\partial_{\mu,3}e^{ik_3x_3}\widetilde{\phi}(k_3)$$

Taking the derivatives:

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{\delta}{\delta\widetilde{\phi}(k_1)}\frac{\delta}{\delta\widetilde{\phi}(k_2)}\frac{\delta}{\delta\widetilde{\phi}(k_3)}\frac{1}{2}g\int d^4k_1d^4k_2d^4k_3e^{ik_1x_1}\widetilde{\phi}(k_1)e^{ik_2x_2}\widetilde{\phi}(k_2)e^{ik_3x_3}\widetilde{\phi}(k_3)$$

To avoid ambiguity, let's change the dummy variable inside the integrand:

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{\delta}{\delta\widetilde{\phi}(k_1)}\frac{\delta}{\delta\widetilde{\phi}(k_2)}\frac{\delta}{\delta\widetilde{\phi}(k_3)}\frac{1}{2}g\int d^4k_1'd^4k_2'd^4k_3'e^{ik_1'x_1}\widetilde{\phi}(k_1')e^{ik_2'x_2}\widetilde{\phi}(k_2')e^{ik_3'x_3}\widetilde{\phi}(k_3')$$

Next, let's take the functional derivatives:

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{1}{2}g\int d^4k_1'd^4k_2'd^4k_3'e^{ik_1'x_1}e^{ik_2'x_2}e^{ik_3'x_3}\delta^4(k_1-k_1')\delta^4(k_2-k_2')\delta^4(k_3-k_3')$$

Doing the integral, we have:

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{1}{2}ge^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}$$

Now all these xs are the same, so:

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{1}{2}ge^{i(k_1+k_2+k_3)x}$$

We'll define the position at the vertex to be zero, causing the plane wave to vanish. Then,

$$\text{V.F.} = -i(k_2\cdot k_3)\frac{1}{2}g$$

Finally, we arbitrarily paired derivatives with fields. These could be paired in any combination, so we must add on the other five combinations. Why add them? Because there are actually six diagrams with six different permutations of the labels, and six different vertex factors. We consider all of them identical (since $x_1 = x_2 = x_3 = x$), so we combine them in one diagram – but then we need to account for the sum in the vertex factor itself. By the way, if the vertex factors were identical, we would just have to account for an overall numerical factor (as indeed happens in problem 10.5).

$$\text{V.F.} = -i(k_2\cdot k_3 + k_3\cdot k_2 + k_1\cdot k_3 + k_3\cdot k_1 + k_2\cdot k_1 + k_1\cdot k_2)\frac{1}{2}g$$

which implies

$$\text{V.F.} = -ig(k_2\cdot k_3 + k_1\cdot k_3 + k_1\cdot k_2)$$

or, perhaps more compactly (since conservation of momentum at the vertex implies that $k_1 + k_2 + k_3 = 0$:

$$\text{V.F.} = \frac{ig}{2}(k_1^2 + k_2^2 + k_3^2)$$

Another advantage of writing it in this way is that if any of the particles are outgoing (meaning that $k_i \to -k_i$ in the vertex factor, the vertex factor will not be affected.

Srednicki 10.5. The scattering amplitudes should be unchanged if we make a field redefinition. Suppose, for example, we have

$$\boldsymbol{\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2}$$

and we make the field redefinition

$$\boldsymbol{\phi \to \phi + \lambda\phi^2}$$

Work out the Lagrangian in terms of the redefined field, and the corresponding Feynman rules. Compute (at tree level) the $\phi\phi \to \phi\phi$ scattering amplitude. You should get zero, because this is a free-field theory in disguise.

The first part of this is not difficult. We make the substitution, noting that $\partial^\mu\phi \to \partial^\mu\phi + 2\lambda\phi\partial^\mu\phi$, and the same for the covariant derivative. Hence,

$$\mathcal{L} = -\frac{1}{2}\left[\partial^\mu\phi + 2\lambda\phi\partial^\mu\phi\right]\left[\partial_\mu\phi + 2\lambda\phi\partial_\mu\phi\right] - \frac{1}{2}m^2(\phi+\lambda\phi^2)^2$$

After simplifying, we write this as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. The $\mathcal{L}_0$ is given by equation 8.4, while $\mathcal{L}_1$ is given by:

$$\mathcal{L}_1 = -2\lambda\phi\partial^\mu\phi\partial_\mu\phi - 2\lambda^2\phi^2\partial^\mu\phi\partial_\mu\phi - m^2\lambda\phi^3 - \frac{1}{2}m^2\lambda^2\phi^4$$

Now we're ready to determine the Feynman rules. This is a real scalar field, so the Feynman rules from ϕ^3 theory still hold up to the vertices. This theory allows both three-point and four-point vertices. We must determine the vertex factors.

The first term has vertex factor $-2i\lambda(k_1^2 + k_2^2 + k_3^2)$, as in problem 10.4.

The second term has vertex factor $-4i\lambda^2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$. The procedure for determining this is the same as in problem 10.4 – but we can cut to the end of the page by replacing the derivatives with ik, accounting for all the permutations (2! from the derivative terms and 2! from the other terms) and ignoring the fields.

The third term has vertex factor $-3!im^2\lambda$ (this is the same as the ϕ^3 theory, modified only for the differences in the constants of the Lagrangian)

The fourth term has vertex factor $-\frac{1}{2}4!im^2\lambda^2$. There are no derivatives, so we just get rid of the fields, and account for the permutations.

Putting this all together, we have the total vertex factors:

- The three-point vertex gets a V.F. of $-2i\lambda(k_1^2 + k_2^2 + k_3^2) - 6im^2\lambda$
- The four-point vertex gets a V.F. of $-4i\lambda^2(k_1^2 + k_2^2 + k_3^2 + k_4^2) - 12im^2\lambda^2$

All other Feynman Rules are unchanged.

Now for the $\phi\phi \to \phi\phi$ scattering process. The good news is that we can use equation

10.14, and our job is to determine τ. We don't even have to draw our own diagrams: the diagrams involving the three-point vertex are those in figure 10.2, and there is only one diagram involving the four-point vertex – the one with four external lines.

What is the value of the diagrams? The four-point vertex diagram gets a factor of 1 for all the external lines – so all that's left is the vertex, which gets a factor of $-4i\lambda^2(k_1^2 + k_2^2 + k_3^2 + k_4^2) - 12im^2\lambda^2 = -4i\lambda^2\left[(k_1^2 + k_2^2 + k_3^2 + k_4^2) + 3m^2\right]$. Rewriting this a bit, we have: $-4i\lambda^2\left[(k_1^2 + m^2) + (k_2^2 + m^2) + (k_3^2 + m^2) + (k_4^2 + m^2) - m^2\right]$. Since all four particles are external, they are on shell, and $k_i^2 = -m^2$. Then, the value of the diagram is simply $4im^2\lambda^2$.

The s-channel diagram has two vertices and a propagator, giving $4i\lambda^2\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]$ $\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]\left[\frac{1}{(k_1+k_2)^2+m^2}\right]$ Adding in the t-channel and u-channel will be the same except for the propagator. Hence, the sum of all four diagrams is:

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]^2\left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_1-k_2')^2+m^2}\right]$$

Now let's distribute the vertex factor.

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[\frac{\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]^2}{(k_1+k_2)^2+m^2} + \frac{\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]^2}{(k_1-k_1')^2+m^2} + \frac{\left[(k_1^2 + k_2^2 + k_3^2) + 3m^2\right]^2}{(k_1-k_2')^2+m^2}\right]$$

Now let's distribute the mass in each of the numerators. In all cases, the vertex consists of one internal line and one propagator. As with the four-point vertex, the external lines are on shell and vanish. So:

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[\frac{\left[k_{int}^2 + m^2\right]^2}{(k_1+k_2)^2+m^2} + \frac{\left[k_{int}^2 + m^2\right]^2}{(k_1-k_1')^2+m^2} + \frac{\left[k_{int}^2 + m^2\right]^2}{(k_1-k_2')^2+m^2}\right]$$

For each channel, we know what the momentum of the internal vertex is:

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[\frac{\left[(k_1+k_2)^2 + m^2\right]^2}{(k_1+k_2)^2+m^2} + \frac{\left[(k_1-k_1')^2 + m^2\right]^2}{(k_1-k_1')^2+m^2} + \frac{\left[(k_1-k_2')^2 + m^2\right]^2}{(k_1-k_2')^2+m^2}\right]$$

which is:

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[(k_1+k_2)^2 + m^2 + (k_1-k_1')^2 + m^2 + (k_1-k_2')^2 + m^2\right]$$

When all of these binomials are expanded out, we use the relation $k_i^2 + m^2 = 0$ to obtain (written in a deliberately strange way):

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[k_1^2 + 2k_1k_1 + 2k_1k_2 - 2k_1k_1' - 2k_1k_2'\right]$$

This first term is simply $-m^2$; the remaining terms can be factored:

$$i\tau = 4im^2\lambda^2 + 4i\lambda^2\left[-m^2 + 2k_1(k_1 + k_2 - k_1' - k_2')\right]$$

The term in parenthesis vanishes by conservation of four-momentum. The remaining terms also cancel, giving:

$$i\tau = 0$$

which makes sense, since there should be no scattering amplitude between two free fields.