Srednicki Chapter 1

QFT Problems & Solutions

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Problem 1.1 (based on Srednicki 1.1). Prove that the Dirac Matrices must be square matrices of dimension 4x4 or higher. Your proof should be rigorous and detailed, though you may assume that you can write the Dirac Matrices in an eigenvalue basis (the proof of this follows from Schur's Lemma, which I do not want to deal with here).

We must disallow four cases:

1. Non-square matrices.
2. 1x1 matrices.
3. 2x2 matrices
4. 3x3 matrices.

I. This is trivial. If a matrix is non-square, then the anticommutator with itself cannot be the identity matrix, but will be the sum of two matrices of different dimensionality.

II. 1x1 matrices are also no good, since such matrices cannot have a nonzero anticommutator. We could use grassman numbers, but the grassman numbers squared give zero, not one as required (more on grassman numbers later).

III. Sredincki suggests that these cannot be 2x2 (since the Pauli Matrices are only three), but it remains to rigorously exclude this possibility. The Pauli Matrices with the identity span the space, but do not all anticommute, so we could use properties of the Clifford Algebra to prove that there cannot be four 2x2 matrices that satisfy the required properties. Instead, I present a more straightforward approach.

Let's start by proving some apparently unrelated claims.

Claim 1: The Dirac Matrices are traceless.
Proof: I will prove the claim here for β, but simply exchange α with β to prove the claim for α.

$$Tr(\beta) = Tr(I\beta)$$

Using Srednicki 1.27, $\alpha_1^2 = I$. Hence,

$$Tr(\beta) = Tr(\alpha_1^2\beta)$$

Using the cyclic property of the trace,

$$Tr(\beta) = Tr(\alpha_1 \beta \alpha_1)$$

Now anti-commute α_1 with β (using Srednicki 1.27):

$$Tr(\beta) = -Tr(\alpha_1^2 \beta)$$

$$Tr(\beta) = -Tr(\beta) \implies Tr(\beta) = 0$$

$\square$

Claim 2: The Dirac Matrices have determinants of 1 or -1.
Proof: (as usual, switch α with β to prove for α. Both give the identity when squared, per the anti-commutation relations).

$$\beta^2 = 1$$

$$det(\beta^2) = det(1)$$

$$det(\beta)det(\beta) = 1$$

$$det\beta = 1 \text{ or } -1$$

$\square$.

Claim 3: The Dirac Matrices have eigenvalues of modulus one.
Proof: Now we'll work in an eigenvalue basis. In this basis we have a diagonal matrix β with determinant of modulus 1 and $\beta^2 = I$, which is obviously orthogonal. ($\beta^T\beta = \beta^2 = I$). It is well-established (see for example Ma & Gu page 5) that an orthogonal matrix must have eigenvalues of modulus one. As usual, this holds for α as well as β. $\square$

Claim 4: The sum of two matrices A and B is Hermitian if and only if A and B are.
Proof: This doesn't really need to be proved, its a mathematical fact that we can assume. But I can't find a good proof of it anywhere, so I will prove it myself. Let's write the two parts of the Dirac Hamiltonian as A and B. Then,

$$H = A + B$$

Now let's write this is Hermitian and non-Hermitian parts. Then:

$$H = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) + \frac{1}{2}(B + B^\dagger) + \frac{1}{2}(B - B^\dagger)$$

The non-Hermitian parts must cancel for the Hamiltonian to be Hermitian. Then,

$$A - A^\dagger = B - B^\dagger$$

Now let's write out $H^\dagger$

$$H^\dagger = \frac{1}{2}(A + A^\dagger) - \frac{1}{2}(A - A^\dagger) + \frac{1}{2}(B + B^\dagger) - \frac{1}{2}(B - B^\dagger)$$

This gives:

$$H - H^\dagger = A - A^\dagger + B - B^\dagger = 0$$

Where the last equality follows because H is Hermitian. Solving this equation, we have

$$A^\dagger - A = B - B^\dagger$$

Equating our two results, we have

$$(B - B^\dagger) = -(B - B^\dagger) \implies B - B^\dagger = 0$$

Hence, B is Hermitian, and so, obviously, is A. $\square$

Claim 5: If AB is Hermitian and A, B commute and B is Hermitian, then A must be Hermitian as well:
Proof:

$$H = AB$$

Expanding into Hermitian and skew-Hermitian parts:

$$H = \left[\frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)\right]\frac{1}{2}(B + B^\dagger)$$

$$H = \frac{1}{2}\left[(A + A^\dagger)(B + B^\dagger) + (A - A^\dagger)(B + B^\dagger)\right]$$

Taking the Hermitian Conjugate, and using commutation:

$$H = H^\dagger = \frac{1}{2}\left[(A + A^\dagger)(B + B^\dagger) - (A - A^\dagger)(B + B^\dagger)\right]$$

Hence,

$$(A - A^\dagger)(B + B^\dagger) = 0$$

B is not zero, so $A = A^\dagger$, and A is Hermitian as well. $\square$

Claim 6: The Dirac Matrices are Hermitian.
Proof: Using claim 4, the two terms of the Hamiltonian are Hermitian. Pulling out the constants, β is clearly Hermitian. For α, we must use claim 5: $\mathbf{p}$ is an observable and so is Hermitian; further this (diagonal) operator will certainly pass through the constant Dirac matrices. Hence claim 5 applies, and the α is Hermitian as well. $\square$

Now we can prove that the 2x2 matrices won't work. Using claim 1 and claim 2, we see that our 2x2 matrices must have the form

$$\begin{pmatrix} a & b \\ \frac{a^2-1}{b} & -a \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ \frac{a^2+1}{b} & -a \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Where the first two options are the most general matrices with no trace and a determinant of plus or minus one, and the third option is what happens if b = 0 (since b cannot be zero in the other two matrices). Let's call these matrices A, B, and C for clarity.

B is no good since squaring it gives -I, rather than I. As a result, the four matrices have to come from A or C, so at least three matrices must be of the form of A.

Is it possible to get three matrices of the form A that anticommute? One matrix (A1) can be chosen at will. The second matrix (A2) must be constructed to anticommute with A1. This will occur if the following constraint is obeyed:

$$2a_1a_2 + \frac{b_1}{b_2} + \frac{b_2}{b_1} - \frac{a_2^2 b_1}{b_2} - \frac{a_1^2 b_2}{b_1} = 0$$

This is just one constraint, so A2 is possible. Similarly, A3 must commute with A1 (one constraint) and A2 (one constraint), so it is possible as well (two constraints with two degrees of freedom). However, there cannot be four A matrices, since we would have three constraints and two degrees of freedom. So our only option is to have A1, A2, A3 and C.

Is this possible? Well, we never enforced the condition that the A's will anticommute with the C's. We don't have any extra degrees of freedom, so this is possible if and only if we can enforce this condition without imposing any new constraints. However, A1 and C only anticommute if a_1 is zero, a condition which is not satisfied by, and nor does it satisfy, any of the other constraints. Hence, it is not possible, as A1, A2 and A3 cannot simultaneously anticommute with C and each other.

IV. Using claim 6, the eigenvalues must be real. Claim 3 tells us that they must be plus or minus one. To have the trace equal to zero (claim 1), the sum of the eigenvalues must be equal to zero, which is only possible when the dimensionality is even. So, 3x3 matrices are no good.

Srednicki 1.2. Show that the Hamiltonian and wave function in non-relativistic field theory are equivalent to that in quantum mechanics. In other words, use the Hamiltonian of eq. (1.32), to show that the state defined in (1.33) obeys the Schrödinger Equation if and only if the wave function obeys (1.30). Do this for fermions and bosons.

We'll treat each term of the Hamiltonian separately. We'll also write ∇^2 as ∇_x^2 to indicate the variable of differerentiation. Then, we have:

$$H_1|\psi\rangle = \int d^3x d^3x_1 \dots d^3x_n a^\dagger(x) \left(-\frac{\hbar^2}{2m}\nabla_x^2 + U(x)\right) a(x)\psi(x_1,\dots,x_n,t)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Remember that x is just a label, no longer an operator. Hence, the only thing that might have difficult commutation relations are the a's. Hence,

$$H_1|\psi\rangle = \int d^3x d^3x_1 \dots d^3x_n \psi(x_1,\dots,x_n,t) \left(-\frac{\hbar^2}{2m}\nabla_x^2 + U(x)\right) a^\dagger(x)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Let's consider these operators at the end. We have:

$$a^\dagger(x)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

(Anti)-commutating the a(x), we have:

$$a^\dagger(x)\left(\delta^3(x-x_1) \pm a^\dagger(x_1)a(x)\right)\dots a^\dagger(x_n)|0\rangle$$

with the top sign for bosons and the bottom sign for fermions.

It follows that we'll have n terms, since the last terms will have $a(x)$ annihilating the vacuum. Each term will have the form $\pm\delta(x-x_i)a^\dagger(x)a^\dagger(x_1)\dots a^\dagger(x_{i-1})a^\dagger(x_{i+1})\dots a^\dagger(x_n)$, with the even terms negative.

Now we must (anti)-commute $a^\dagger(x)$ to the position where $a^\dagger(x_i)$ used to be. The bosonic terms will all commute. The fermionic terms will anticommute, resulting in a plus sign for all odd terms (for example, the first term will require no anti-commutation), and a minus sign for all even terms. Hence, the minus signs cancel, and we end up with n terms of the form:

$$\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

As a result:

$$H_1|\psi\rangle = \sum_{i=1}^{n}\int d^3x d^3x_1\dots d^3x_n\psi(x_1,\dots,x_n,t)\left(-\frac{\hbar^2}{2m}\nabla_x^2+U(x)\right)\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Now we use the delta function:

$$H_1|\psi\rangle = \sum_{i=1}^{n}\int d^3x_1\dots d^3x_n\psi(x_1,\dots,x_n,t)\left(-\frac{\hbar^2}{2m}\nabla_{x_i}^2+U(x_i)\right)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

which of course gives:

$$H_1|\psi\rangle = \sum_{i=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_{x_i}^2+U(x_i)\right)|\psi\rangle$$

Now for the second term. We'll start by observing:

$$[a^\dagger(x),a^\dagger(y)a(y)] = [a^\dagger(x),a^\dagger(y)]a(y)+a^\dagger(y)[a^\dagger(x),a(y)]$$

$$[a^\dagger(x),a^\dagger(y)a(y)] = [a^\dagger(x),a^\dagger(y)]a(y)-a^\dagger(y)[a(y),a^\dagger(x)]$$

For bosons, the first term vanishes, the second term gives $-a^\dagger(y)\delta(x-y)$. For fermions, we must rewrite these as anticommutators. To do this, we must subtract $a^\dagger(y)a^\dagger(x)a(y)$ for the first term, and add $a^\dagger(y)a^\dagger(x)a(y)$ for the second term, so these cancel. The result is the same.

Now we have:

$$H_2|\psi\rangle = \frac{1}{2}\int d^3xd^3yd^3x_1\dots d^3x_nV(x-y)\psi(x_1,\dots,x_n,t)a^\dagger(x)a^\dagger(y)a(y)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

The annihilation and creation operators are:

$$a^\dagger(x)a^\dagger(y)a(y)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Using the commutation relation above:

$$a^\dagger(y)a(y)a^\dagger(x)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle - a^\dagger(y)\delta(x-y)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

This first term has the same form as the operator expression in H_1, so we can quote the result. The operators become:

$$a^\dagger(y)a(y)\sum_{i=1}^{n}\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle - a^\dagger(y)\delta(x-y)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Using the same trick again, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\delta(x-x_i)\delta(y-x_j)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle - a^\dagger(y)\delta(x-y)a(x)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Now if x = y, we're looking at the particle interacting with itself, which will be zero. So this second term vanishes.

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\delta(x-x_i)\delta(y-x_j)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

Now let's plug into the expression for the Hamiltonian:

$$H_2|\psi\rangle = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int d^3x_1\dots d^3x_n V(x_i-y_j)\psi(x_1,\dots,x_n,t)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

The factor of one-half is designed to eliminate double-counting. Writing this differently:

$$H_2|\psi\rangle = \sum_{i=1}^{n-1}\sum_{j=1}^{n}\int d^3x_1\dots d^3x_n V(x_i-y_j)\psi(x_1,\dots,x_n,t)a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle$$

By the way, you may be wondering why we don't have to account for self-interaction terms in this piece of the Hamiltonian. The answer is that such terms will not contribute to the sum. It is not necessary to explicitly disallow this possiblity (though we could if we wanted to). In any case, we use the definition of $|\psi\rangle$ to find:

$$H_2|\psi\rangle = \sum_{i=1}^{n-1}\sum_{j=1}^{n}V(x_i-y_j)|\psi\rangle$$

Now, the Schrödinger equation is:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

The left hand side is the same in both quantum field theory and quantum mechanics. The right hand side is given by (in field theory) Srednicki 1.32 and 1.33. Above, we have rewritten this right hand side as:

$$i\hbar\frac{\partial\psi}{\partial t} = \sum_{i=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_{x_i}^2 + U(x_i)\right)|\psi\rangle + \sum_{i=1}^{n-1}\sum_{j=1}^{n}V(x_i-y_j)|\psi\rangle$$

which implies:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\sum_{i=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_{x_i}^2 + U(x_i)\right) + \sum_{i=1}^{n-1}\sum_{j=1}^{n}V(x_i-y_j)\right]|\psi\rangle$$

exactly as it is written in quantum mechanics. Hence, field theory is equivalent to quantum mechanics in the non-relativistic limit.

Srednicki 1.3. Show explicitly that the non-relativistic Hamiltonian (1.32) and the Number Operator commute.

We'll start by proving that $[N(x), a^\dagger(y)] = a^\dagger(y)$.

$$[N(x), a^\dagger(y)] = [\int d^3x a^\dagger(x)a(x), a^\dagger(y)]$$

$$[N(x), a^\dagger(y)] = \int d^3x \left([a^\dagger(x), a^\dagger(y)]a(x) + a^\dagger(x)[a(x), a^\dagger(y)]\right)$$

For bosons, we're done. The first term vanishes; the second commutator becomes a delta function which kills the integral, leaving us with $a^\dagger(y)$. For fermions, we have to switch the commutators to anticommutators.

$$[N(x), a^\dagger(y)] = \int d^3x \left(\{a^\dagger(x), a^\dagger(y)\}a(x) + a^\dagger(x)\{a(x), a^\dagger(y)\} - 2a^\dagger(y)a^\dagger(x)a(x) - 2a^\dagger(x)a^\dagger(y)a(x)\right)$$

Anticommuting the last term gives

$$[N(x), a^\dagger(y)] = \int d^3x \left(\{a^\dagger(x), a^\dagger(y)\}a(x) + a^\dagger(x)\{a(x), a^\dagger(y)\} - 2a^\dagger(y)a^\dagger(x)a(x) + 2a^\dagger(y)a^\dagger(x)a(x)\right)$$

Hence,

$$[N(x), a^\dagger(y)] = \int d^3x \left(\{a^\dagger(x), a^\dagger(y)\}a(x) + a^\dagger(x)\{a(x), a^\dagger(y)\}\right)$$

We're done. The first anticommutator vanishes, the second gives a delta function which kills the integral, leaving us with $a^\dagger(y)$. In the same way, we can show that $[N(x), a(y)] = -a(y)$.

Now we're ready to approach a more general commutator:

$$[N(x), a^\dagger(y_1)\ldots a^\dagger(y_n)a(z_1)\ldots a(z_m)] = [N(x), a^\dagger(y_1)]a^\dagger(y_2)\ldots a^\dagger(y_n)a(z_1)\ldots a(z_m) +$$

$$a^\dagger(y_1)[N(x), a^\dagger(y_2)]\ldots a^\dagger(y_n)a(z_1)\ldots a(z_m) + \ldots + a^\dagger(y_1)\ldots a^\dagger(y_n)a(z_1)\ldots [N(x), a(z_m)]$$

Inserting the value of the commutators we defined above, we get:

$$[N(x), a^\dagger(y_1)\ldots a^\dagger(y_n)a(z_1)\ldots a(z_m)] = (n-m)a^\dagger(y_1)a^\dagger(y_2)\ldots a^\dagger(y_n)a(z_1)\ldots a(z_m)$$

So, the commutator will vanish so long as there are the same number of $a^\dagger$'s as a's. This is true for every term in the Hamiltonian. (Don't be concerned about stuff like the gradient – that will commute with creation and annihilation operators. Why? Because position is no longer an operator, just a label. Moreover, the creation/annihilation operators do not "live" in position space). Hence,

$$[N, H] = 0$$

as expected.

Problem 1.4. Prove that the symmetric part (for fermions) and the anti-symmetric part (for bosons) of the following expression (Srednicki 1.33) vanishes:

$$|\mathbf{\Psi(x, t)}\rangle = \int \mathbf{d^3x_1 d^3x_2}\psi(\mathbf{x_1, x_2})\mathbf{a^\dagger(x_1)a^\dagger(x_2)}|0\rangle$$

x_1 and x_2 are dummy variables, so we can switch them without consequence:

$$|\Psi(\mathbf{x},t)\rangle = \int d^3x_1 d^3x_2 \psi(\mathbf{x}_2,\mathbf{x}_1) a^\dagger(\mathbf{x}_2) a^\dagger(\mathbf{x}_1)|0\rangle$$

Let's do fermions first. We can divide ψ into symmetric and anti-symmetric parts. Now we'll show that the symmetric part will vanish:

$$|\Psi(\mathbf{x},t)\rangle_{symm} = \int d^3x_1 d^3x_2 \psi(\mathbf{x}_2,\mathbf{x}_1)_{symm} a^\dagger(\mathbf{x}_2) a^\dagger(\mathbf{x}_1)|0\rangle$$

$$|\Psi(\mathbf{x},t)\rangle_{symm} = \int d^3x_1 d^3x_2 \psi(\mathbf{x}_1,\mathbf{x}_2)_{symm} a^\dagger(\mathbf{x}_2) a^\dagger(\mathbf{x}_1)|0\rangle$$

Using the anti-commutation relations for fermions:

$$|\Psi(\mathbf{x},t)\rangle_{symm} = -\int d^3x_1 d^3x_2 \psi(\mathbf{x}_1,\mathbf{x}_2)_{symm} a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2)|0\rangle$$

$$|\Psi(\mathbf{x},t)\rangle_{symm} = -|\Psi(\mathbf{x},t)\rangle_{symm} \implies |\Psi(\mathbf{x},t)\rangle_{symm} = 0$$

Now for bosons. As before, we'll divide ψ into symmetric and anti-symmetric parts. Now we'll show that the anti-symmetric part will vanish:

$$|\Psi(\mathbf{x},t)\rangle_{anti} = \int d^3x_1 d^3x_2 \psi(\mathbf{x}_2,\mathbf{x}_1)_{anti} a^\dagger(\mathbf{x}_2) a^\dagger(\mathbf{x}_1)|0\rangle$$

$$|\Psi(\mathbf{x},t)\rangle_{anti} = -\int d^3x_1 d^3x_2 \psi(\mathbf{x}_1,\mathbf{x}_2)_{anti} a^\dagger(\mathbf{x}_2) a^\dagger(\mathbf{x}_1)|0\rangle$$

Using the commutation relations for bosons:

$$|\Psi(\mathbf{x},t)\rangle_{anti} = -\int d^3x_1 d^3x_2 \psi(\mathbf{x}_1,\mathbf{x}_2 a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2)|0\rangle$$

$$|\Psi(\mathbf{x},t)\rangle_{anti} = -|\Psi(\mathbf{x},t)\rangle_{anti} \implies |\Psi(\mathbf{x},t)\rangle_{anti} = 0$$

Hence, $|\Psi(\mathbf{x},t)\rangle$ must be symmetric for bosons and antisymmetric for fermions.

Problem 1.5. Prove that the Klein-Gordon equation is consistent with relativity but not with quantum mechanics. The first part was done in the text, but go through the derivation again here.

The trick is to write the Klein-Gordon equation in useful notation:

$$(-\partial^2 + \frac{m^2c^2}{\hbar^2})\psi(x) = 0$$

To see that this is correct, we'll write as (supressing the x label):

$$-\frac{1}{c^2}\hbar^2\frac{\partial^2\psi}{\partial t^2} + \hbar^2\nabla^2\psi - m^2c^2\psi = 0$$

which is the Klein-Gordon equation in the form presented before. Now, we'll write the Klein-Gordon equation from the perspective of someone else, in a different frame:

$$(-\overline{\partial}^2 + \frac{m^2c^2}{\hbar^2})\overline{\psi}(\overline{x}) = 0$$

The two observers must agree on the observable, so $\psi(x) = \overline{\psi}(\overline{x})$.

$$(-\overline{\partial}^2 + \frac{m^2c^2}{\hbar^2})\psi(x) = 0$$

Now for the $\overline{\partial}^2$. This is done in Srednicki 1.21, but let's explain it here. First, by definition, $\overline{\partial}^2 = g_{\mu\nu}\overline{\partial}^\mu\overline{\partial}^\nu$. Now, these two operators, ∂ and $\overline{\partial}$, are related by a Lorentz Transformation (that's what special relativity is), so we can write $\overline{\partial^2}$ as: $\overline{\partial}^2 = g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma\partial^\rho\partial^\sigma$. The fact that the interval is invariant means that g will not be affected by Lorentz Transformations, so $\overline{\partial}^2 = g_{\rho\sigma}\partial^\rho\partial^\sigma = \partial^2$. Hence, the two observers agree on the Klein-Gordon equation, which *is* therefore consistent with relativity.

As for quantum mechanics, we note that the norm of a state is given by (we'll work in position space):

$$\langle\psi \mid \psi\rangle = \int d^3x\psi^*\psi$$

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = \int d^3x[\psi^*\frac{d\psi}{dt} + \frac{d\psi^*}{dt}\psi]$$

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = \frac{1}{i\hbar}\int d^3x[\psi^*(-\frac{\hbar^2}{2m})\nabla^2\psi + (\frac{\hbar^2}{2m}\nabla^2\psi^*)\psi]$$

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = \frac{\hbar}{2mi}\int d^3x[-\psi^*\nabla^2\psi + (\nabla^2\psi^*)\psi]$$

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = \frac{\hbar}{2mi}\int d^3x\nabla\cdot[-\psi^*\nabla\psi + (\nabla\psi^*)\psi]$$

Now we integrate by parts, switching the derivative to the constant term, which of course vanishes. We're left with a boundary term, which also vanishes. Hence, the Schrödinger equation has a constant norm, ie probability is conserved.

When we try this argument for the Klein-Gordon equation, we see that $-\hbar^2\frac{\partial\psi}{\partial t} = -\int dt(-\hbar^2c^2\nabla^2 + m^2c^4)\psi$. Then, we have:

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = -\int d^3x[\psi^*(-\hbar^2)\frac{d\psi}{dt} + -\hbar^2\frac{d\psi^*}{dt}\psi]$$

$$\frac{d}{dt}\langle\psi \mid \psi\rangle = \int d^3x[\psi^*\int dt(-\hbar^2c^2\nabla^2 + m^2c^4)\psi + \int dt(-\hbar^2c^2\nabla^2 + m^2c^4)\psi^*\psi]$$

These second and fourth terms integrate to give us something that is time dependent. They can't cancel with anything, since mass does not enter the equation in any other term. Hence, the norm of a state is not time-dependent, and the Klein-Gordon equation does not conserve probability. It is therefore *inconsistent* with quantum mechanics.