QFT

Unit 9: The Path Integral for the Interacting Field Theory

Overview

- This is it! We’ll add an interaction term to our Lagrangian and compute Z(J).
  - From Z(J) we can calculate correlation functions
  - From correlation functions we can use the LSZ formula to calculate transition amplitudes.
  - In fact, Z(J) is the gift that keeps on giving! We’ll find other uses for it later.

- Along the way, we’ll discover Feynman Diagrams!

The φ^3 Theory Lagrangian

- We'll assume that our interacting Lagrangian looks like this:
$$\mathcal{L} = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 + Y\phi$$
- In chapter 5, we agreed to renormalize this according to the following scheme:
$$\langle 0|\phi(x)|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{-ikx}$$
- Notice that we have 4 constraints (the two above, m = mass, g = coupling constant) for 4 unknowns.
- This is called φ^3 theory. It is not physically realistic because as φ gets arbitrarily large, L gets arbitrarily negative. But, it works well enough in perturbation theory to be an excellent example for us.

The φ³ Theory Path Integral

- We would like to find Z(J).

$$Z(J) = \langle 0|0\rangle_J = \int \mathcal{D}\phi\ e^{i\int d^4x[\mathcal{L}_0+\mathcal{L}_1+J\phi]}$$

  where $L = L_0 + L_1$

- Remember that J is an arbitrary, external source. It's useful because it allows us to take functional derivatives to calculate correlation functions – but we always set it to zero before getting a physical answer.

- We'll break the Lagrangian into two parts, and pull the second one out of the integral – exactly as we did in equation 6.22.

$$Z(J) = e^{i\int d^4x\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\phi\ e^{i\int d^4x[\mathcal{L}_0+J\phi]}$$

The φ³ Theory Path Integral, cntd.

$$Z(J) = e^{i\int d^4x \mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(x)})} \int \mathcal{D}\phi \; e^{i\int d^4x[\mathcal{L}_0 + J\phi]}$$

- Does this integral look familiar? It’s the free-field path integral! Let’s call the result Z_0. Then,

  $$Z(J) \propto e^{i\int d^4x \mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(x)})} Z_0(J)$$

  where

  $$Z_0(J) = exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]$$

  the result from last time.
  - Note that the equality has become a proportionality, since invoking our “epsilon trick” to determine $Z_0(J)$ destroyed the normalization.
  - Free fields shouldn’t transition, so we can renormalize by hand by requiring $Z_0(0) = 1$.

The φ³ Theory Lagrangian, again

- Since L_0 must be the free field Lagrangian, it follows that L_1 must be:

$$\mathcal{L}_1 = \frac{1}{6} Z_g g \phi^3 - \frac{1}{2}(Z_\phi - 1)\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}(Z_m - 1) m^2 \phi^2 + Y\phi$$

- These last three terms are just the renormalization requirements. Let's count them separately. Then, our Lagrangian has three components:

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L}_1 = \frac{1}{6} Z_g g \phi^3$$

$$\mathcal{L}_{ct} = -\frac{1}{2}(Z_\phi - 1)\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}(Z_m - 1) m^2 \phi^2 + Y\phi$$

Organizing $Z_1(J)$

- There are 2P sources, but 3V of them are killed by the derivatives. So, the number of sources for a given term is E = 2P – 3V
- There are P propagators
- The phase factor is i^{P-2V}
- Let's expand one term:
  - after 0 derivatives: 1 term
  - after 1 derivative: (2P) terms
  - after 2 derivatives: (2P) * (2P-1) terms
  - after 3 derivatives: (2P) * (2P-1) * (2P-2) terms
  - after 3V derivatives: (2P) * (2P-1) * … * (2P – 3V + 1) terms
  - ie, a given term above will have (2P)! / (2P – 3V)! terms after expansion
- But, many of these terms are equivalent. We'd like to know how many terms have a given V and E.

$$Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{iZ_g g}{6}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V$$
$$\times \sum_{P=0}^{\infty}\left[\frac{i}{2}\int d^4y d^4z J(y)\Delta(y-z)J(z)\right]^P$$

The φ^3 Theory Path Integral, again

$$Z(J) \propto e^{i\int d^4x \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J)$$

$$Z_0(J) = exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]$$

- Let's plug in L_1. Note that the argument of L_1 (phi) is replaced by –i d/dJ. Then, neglecting the counterterms:

$$Z_1(J) \propto exp\left[\frac{i}{6}Z_g g\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right] Z_0(J)$$

- These are just two exponentials, so we'll expand:

$$Z_1(J) \propto \sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{iZ_g g}{6}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V$$

$$\times \sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{i}{2}\int d^4y d^4z J(y)\Delta(y-z)J(z)\right]^P$$

Proto-Feynman Diagrams

- To determine the number of terms with a given E and V, we introduce these proto-Feynman diagrams:
  - The idea is to represent every term with a diagram
  - But we'll actually work in reverse – draw the diagrams and see how many terms correspond to it.
- The rules for drawing are this:
  - $P = \frac{1}{2}(E+3V)$ propagators, represented by a line. These lines include the 1/i factor
  - E sources, represented by filled circles. A filled circle connected to a line segment for a source. This includes the i factor and the integral.
  - Vertices connecting three line segments for $i Z_g g\, d^4x$

Proto-Feynman Diagrams: Example

- Let's write the term with E = 1, V = 1. After simplifying (a lot of messy algebra), the term is:

$$\frac{-iZ_g g}{48}\left[\int d^4x d^4b \Delta(0)\Delta(x-b)J(b) + 23 \text{ other terms}\right]$$

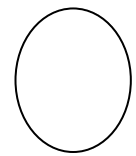
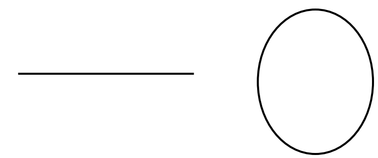
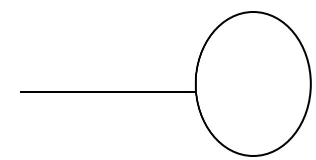
- But, many of these are the same:
  - 24 terms become 4, because the variables that had been different were equated after the functional integrals
  - 4 terms become 1, because the four remaining terms are the same up to a dummy variable, which doesn't matter.

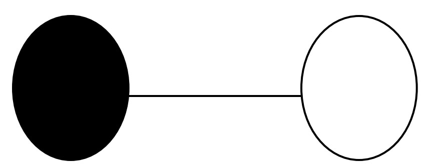
- The result is: $\frac{-iZ_g g}{2}\int d^4x d^4b \Delta(0)\Delta(x-b)J(b)$

- **In general**
  - **There might be more than one surviving term.**
  - **But, terms with other E, V cannot contribute because they'll have different numbers of Js.**

Proto-Feynman Diagrams: Example

- Now let's draw this:

1. $\frac{1}{i}\Delta(0)$ is represented by 
2. $-\Delta(x-b)\Delta(0)$ is represented by 
3. $-\int d^4 x i Z_g g \Delta(x-b)\Delta(0)$ is represented by 
4. $-\int d^4 x d^4 b i Z_g g \Delta(x-b)\Delta(0) J(b)$ is represented by



So this diagram accounts for everything except the factor of 2

Proto-Feynman Diagrams: The Numerical Factor

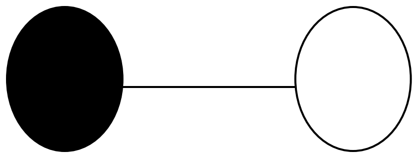
- How can we predict the numerical factor from the diagram?
  - $1/(V!\,P!\,6^V\,2^P)$ from the Taylor Series
  - But, many of the changes between terms won't affect the diagram.
    - We can swap the order of the functional derivatives: $(3!)^V = 6^V$ terms
    - Swap the vertices: $V!$ terms
    - Swap the order of the sources: $(2!)^P = 2^P$ terms
    - Swap the propagators: $P!$ terms
  - These cancel neatly! Numerical factor = 1

- But did we overcount?
  - What if multiple swaps give the same diagram?
  - This tends to happen when the diagram has symmetry.

- So, must divide our numerical factor by the symmetry factor of the diagram.

Proto-Feynman Diagrams: Example, again

- Does our previous diagram have any symmetry?



- Swapping the two “ends” of the curved propagator is the same as swapping those two legs of the vertex. So we overcounted by a factor of 2, and S = 2.

- Mathematically, this is because some derivative permutations are the same as some source permutations.
  - For example, imagine derivatives acting on a, b, c, but not d. Relabeling the derivatives to act on b, c, d (not a) is the same as relabeling the sources to be named b, c, d, a.

Proto-Feynman Diagrams: The Symmetry Factor

- Summarizing: we have assumed that the diagrams can be redrawn in several ways without changing the diagrams:
  - The sources can be swapped:
    - Remember that sources are just combinations of propagators and vertices. Whatever they are, they can be swapped with each other.
    - We may gain an additional symmetry factor when we examine the form of the source.
  - The legs of the vertex can be swapped
  - The vertices themselves can be swapped
  - The end of each propagator can be swapped.
  - The propagators themselves can be swapped.
- If any distinct combination of the above gives the same result, then we have overcounted. We should then remove one of the offending combinations from this list, and divide by the symmetry factor.

General Diagrams

- Consider the case with E = 4, V = 4. There are six diagrams, each with a different symmetry factor.
  - These are all *connected*

- The general diagram is the product of all the connected diagrams.
  - There is an additional symmetry factor, due to inter-diagram transposes. Additionally, a connected diagram can appear more than once. So:

$$D = \frac{1}{S_D} \prod_I (C_I)^{N_I}$$

General Diagrams, Cntd.

- What is the additional symmetry factor?
  - The total diagram will be unchanged only if all the propagators and vertices from one connected diagram replace all the propagators and vertices from another, identical connected diagram.
    - Why? For example, a vertex has three propagators attached. The only way the same vertex can be attached to the same propagators – resulting in an identical diagram that needs to be accounted for in the additional symmetry factor – is if all the vertices and propagators are swapped simultaneously.
- From this, the additional symmetry factor is obviously $N_I!$

$Z_1(J)$

- Did these diagrams help us simplify $Z_1(J)$?
  - Yes. $Z_1(J)$ is the sum of these diagrams, so:

$$\begin{aligned} & Z_1(J) \quad \propto \quad \textstyle\sum_{n_I} D \\ \Longrightarrow \quad & Z_1(J) \quad \propto \quad exp\left(\textstyle\sum_I C_I\right) \end{aligned}$$

  - This is a remarkable result: $Z_1(J)$ is the exponential of the sum of connected diagrams.
  - We can also easily impose normalization $Z_1(0) = 1$. Just omit the vacuum diagrams (those with no sources), so that the exponential will vanish. Hence,

$$Z_1(J) = exp\left[i \sum_{I \neq \text{vacuum}} C_I \right]$$

Correlation Functions

- The whole point of Z(J) is to calculate correlation functions for the LSZ formula, so we should calculate some correlation functions now.
  - Remember that we threw away the counterterms, so we'll have to account for that at some point.

$$\langle 0|\phi(x)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x)} Z_1(J)\bigg|_{J=0} \qquad \langle 0|\phi(x)|0\rangle = \frac{\delta}{\delta J(x)} W_1(J)\bigg|_{J=0}$$

  - This is the sum of all diagrams with one source, with the source removed. In fact, this is the functional derivative of the term we calculated earlier:

$$\langle 0|\phi(x)|0\rangle = \frac{ig}{2}\int d^4y \frac{1}{i}\Delta(x-y)\frac{1}{i}\Delta(y-y) + O(g^3)$$

Correlation Functions, Cntd.

$$\langle 0|\phi(x)|0\rangle = -\frac{ig}{2}\int d^4y\Delta(x-y)\Delta(0) + O(g^3)$$

- Understanding this:
  - There are no terms of order two because one cannot draw a diagram with one source and two vertices.
  - We set Z_g = 1 because Srednicki claims Z_g = 1 + $O(g^2)$

- The problem:
  - Remember our normalization agreement:

$$\langle 0|\phi(x)|0\rangle = 0$$

  - This is obviously not true!

Counterterms

- To fix this, need to add one of our counterterms, $L_{ct,\ partial} = Y\varphi$
  - This introduces a new vertex, one where a phi particle stops. This is represented diagrammatically by an x, which accounts for dy, the integral, Y, and i.

- To order g (one vertex), this introduces one new diagram. We add this diagram to the sum and find that to impose the normalization, we need:

$$Y = \frac{1}{2} i g \Delta(0) + O(g^3)$$

Counterterms, cntd.

$$Y = \frac{1}{2} ig\Delta(0) + O(g^3)$$

- Recall Y is by definition the coefficient of an Hermitian operator; it should therefore be real. Is this real? To see, consider:

$$\Delta(0) = \int \frac{d^3k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon}$$

- So is it real? Maybe. But we have another problem: this integral diverges at large k.
  - Solve this by introducing finite "ultraviolet cutoff" as upper bound of the integral.
    - Does this sound like bull? Only a little. There is good reason to expect that the QFT should look different above a certain value. We assume only that it looks different in such a way that the contribution to the propagator is negligible.

Counterterms, cntd.

- We want this integral to be Lorentz Invariant, so we implement the ultraviolet cutoff in a more subtle way (eq. 9.22). The result is:

$$\Delta(0) = \frac{i}{16\pi^2}\Lambda^2$$

- This is great: convergent and imaginary!
- Can also let $\Lambda \rightarrow \infty$, since the normalization condition is still satisfied.
  - The physical meaning of Λ will have to be addressed at some point, though.
  - Having Y infinite may be disturbing, but we'll see that such terms will cancel when measuring physical properties.

Counterterms and Tadpoles

- Same procedure at higher orders: set power of Y as necessary to keep the vacuum expectation value of φ set at zero.

- So, we have an infinite number of one-source diagrams, all of which sum to zero.
  - Let's now replace the source by another diagram. This diagram will act as the source. The sum is still zero!
  - So, we can ignore all these diagrams, since they'll just be cancelled by the Y counterterm.
  - "The rule is this: ignore all diagrams that, when a single line is cut, falls into two parts, one of which has no sources." These are called tadpoles.

More Counterterms

- Adding other terms to the Lagrangian gives us another exponential. We'll integrate by parts and define A, B = Z_φ - 1, Z_m - 1. The result is:
$$Z(J) = exp\left[-\frac{i}{2}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)(-A\partial_x^2 + Bm^2)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]e^{i\sum_{\text{non-vacuum}} C_I}$$
where the diagrams are written with the rules previously discussed.

- This gives a new vertex at which two lines meet. The vertex factor is: $(-i)\int d^4x(-A\partial_x^2 + Bm^2)$
  - The partial acts on either (but not both) of the propagators in the vertex.

Conclusions

- Z(J) = exp[i W(J)], where W(J) is the sum of all connected diagrams. The diagrams must follow the rules discussed before (no tadpoles, two types of vertices, etc.)
- Of course, all this is just for φ^3 theory – other theories will have different rules for drawing the diagrams (see problems 9.2, 9.3)