QFT

Unit 8: Path Integrals for Free Field Theory

Overview

- In this short section, we apply our analysis of path integrals to free-fields.
- As a result, we'll be able to solve the correlation functions that occur in the LSZ formula, for the case of free fields.
  - Of course, free fields never undergo transitions, so this doesn't help us all that much!

The ground state to ground state transition amplitude

- The Hamiltonian is:

$$\mathcal{H}_0 = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

  - No black magic here! This is the Hamiltonian that we posited in chapter 1, checked for reasonableness in problem 1.2, and canonically quantized in chapter 3.

- The corresponding Lagrangian is (derived in 3.9):

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$$

- The fields do not interact, but there can still be an external force fq as with the harmonic oscillator. In the quantized theory, though, we take q → φ and f → J (the *source* of the field). Then:

$$Z_0(J) = \langle 0|0\rangle_J = \int \mathcal{D}\phi \; e^{i\int d^4x[\mathcal{L}_0 + J\phi]}$$

The Propagator, and Z(J)

- As before, we now do a lot of math to simplify the integral. The method is the same as before, with minor changes
  - 4D path integrals this time
  - The Green's function for the Klein-Gordon equation is called the *propagator*.

$$\Delta(x-x') = i\int \widetilde{dk} e^{ik\cdot(x-x')-i\omega|t-t'|}$$

- The result is:

$$Z(J) = exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]$$

Correlation Functions!

- By taking functional derivatives with J, we can find the ground-state expectation values of the φ operators.
  - These are the correlation functions!
  - Expressed in terms of propagators
  - Result is called Wick's Theorem:

$$\langle 0|T\phi(x_1)\ldots\phi(x_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} \Delta(x_{i_1}-x_{i_2})\ldots\Delta(x_{i_{2n-1}}-x_{i_{2n}})$$