QFT

Chapter 70: Group Representations

Representation Theory

- We've discussed transformations before, for example the U(1) transformation on a scalar field, which should leave the Lagrangian invariant:

$$\phi \longrightarrow e^{i\alpha}\phi$$

- In general, this α might not be a scalar, and might not even by commutative.
  - But it can't be just anything. We still need our Lagrangian to be gauge invariant if this is to be useful.

- We can decompose our α into a particular representation, defined by a number of generator matrices T_R.
  - The number of generator matrices is equal to the number of degrees of freedom. These follow commutation relations that define the structure constants.
  - We can, if we desire, change to a different set of traceless basis matrices that follows the same commutation relations.
    - The original set is called "fundamental"
    - The new set is called the D(R) representation, where D(R) is the dimension.
    - For example, taking the complex conjugate of the fundamental representation yields the complex conjugate representation (assuming the fundamental representation is not real).

Adjoint Representation

- The adjoint representation is obtained by setting the generator matrices equal to the structure constants via

$$(T^a_A)^{bc} = -i f^{abc}$$

- This is useful because
  - The generator matrices are Hermitian
  - The adjoint representation is real.
  - The dimension D(A) is equal to the number of generators.

Index and Casimir

- Two useful numbers describe a representation:
  - The index is if multiply any two generators together and take the trace; this should give the same constant.
  - The quadratic Casimir is if you multiply a matrix times itself, that should give the same constant every time (multiplied by the identity).

In some sense, these are both measures of normalization, however the Casimir is dimension-independent.

Direct Sums & Products

- Two representations can be combined into one big representation by the direct sum. This is equivalent to having one large matrix with two diagonal blocks.
  - Irreducible representations cannot be split into diagonal blocks.

- Fields can be written with an index like iI, where i is in one representation and I is the other representation. These two letters together form one index.

Invariant Symbols

- Some symbols are invariant, in that transforming each index gets us back where we started.
  - In a direct product, these claim one index for each represntation.
  - These symbols are a singlet, with a zero generator matrix (as it is invariant). For example, delta or varepsilon can be good invariants.
  - These normally occur in the direct product of two fields, in which case the direct product can be decomposed into a singlet plus everything else (which might decompose further).