QFT

Unit 7: Path Integrals for Harmonic Oscillators

Overview

- In this short section, we apply our analysis of path integrals to harmonic oscillators, obtaining some useful formulas.

- We'll also encounter Green's Functions for the first time.

Ground State to Ground State Transition Amplitude

- The Hamiltonian is:

$$H(P,Q) = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

- So the vacuum-to-vacuum transition amplitude, with an external force, is (from last time):

$$\langle 0|0\rangle_f = \int \mathcal{D}p\mathcal{D}q \; exp \; i\int_{-\infty}^{\infty} dt \, [p\dot{q} - (1-i\varepsilon)H + fq]$$

- Now we do two things:
  - We note that: $H \to (1-i\varepsilon)H$ is equivalent to:

    $$m^{-1} \to (1-i\varepsilon)m^{-1} \qquad m\omega^2 \to (1-i\varepsilon)m\omega^2$$

  - We rewrite in the Lagrangian Formalism

Ground State to Ground State Transition Amplitude

- The result is:

$$\langle 0|0\rangle_f = \int \mathcal{D}q \; i\int_{-\infty}^{\infty} dt \left[\frac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)m\omega^2 q^2 + f\,q\right]$$

- The rest of the section is just evaluating these integrals:
  - Take Fourier transforms
  - Integrate over t and E
  - Change integration variables
  - Observe that the path integral gives the transition amplitude in a vacuum, which is one.
  - Switch to time-variables and invoke a Green's function

Ground State to Ground State Transition Amplitude, Cntd.

- None of that math should be new, except maybe Green's functions, which we'll talk about next.

- The result is:

$$\langle 0|TQ(t_1)\ldots Q(t_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i1}-t_{i2})\ldots G(t_{i(2n-1)}-t_{i(2n)})$$

Green's Functions

- The equation of motion for our situation is:

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\psi(q,t) = f(t)$$

- It's simpler to solve this with:

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)G(t-t') = \delta(t-t')$$

- Then, the solution is given by:

$$\psi(t) = \int G(t-t')f(t')dt'$$