QFT

Unit 6: Path Integrals in Quantum Mechanics

Overview

- We introduce path integrals, which will be needed to solve the correlation functions in the LSZ formula.

- This section is just about path integrals as they appear in Quantum Mechanics. Along the way, we encounter:
  - The Heisenberg & Schrödinger pictures
  - The Campbell-Baker-Hausdorff formula
  - Functional Derivatives

The Schrödinger Picture

- The Schrödinger picture is the “perspective” that the states are constant and the operators are time-dependent.

- We want to know that the probability that a particle goes from q’,t’ to q’’,t’’.

- In the Schrödinger picture, we evaluate this by time-evolving the initial state:

$$\langle q''|e^{-iH(t''-t')}|q'\rangle$$

The Heisenberg Picture

- The Heisenberg picture is the "perspective" that the operators are constant and the states are time-dependent.
- This time the states of the system are defined at particular times, so the transition probability is:

$$\langle q'', t'' | q', t' \rangle$$

- By the way, the kets are related by:

$$|q, t\rangle = e^{iHt} |q\rangle$$

Heisenberg + Schrödinger

- We rewrite the Schrödinger picture, then use the last "observation" from the Heisenberg picture:

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \langle q''|e^{-iHt''}e^{iHt'}|q'\rangle = \langle q'',t''|q',t'\rangle$$

- We can write this by splitting up the exponential into many terms and inserting many complete sets of position eigenstates:

$$\langle q'',t''|q',t'\rangle = \int_{-\infty}^{\infty}\prod_{j=1}^{N} dq_j \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle\ldots\langle q_1|e^{-iH\delta t}|q'\rangle$$

where $\sum \delta t = t'' - t'$

Evaluate the instantaneous bra-kets

- To evaluate this, we need:

$$\langle q_2 | e^{-iH\delta t} | q_1 \rangle$$

- Now we use Baker-Campbell-Hausdorff to expand the exponential, insert a complete set of momentum eigenstates, and perform the integrals (Srednicki eqns. 6.4, 6.5) to achieve

$$\langle q_2 | e^{-iH\delta t} | q_1 \rangle = \int \frac{dp_1}{2\pi} e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}$$

- For a non-free particle, we might have both P and Q, rather than just P. We adopt Weyl ordering, which means that (using the midpoint rule):

$$H(P,Q) = \int \frac{dx}{2\pi}\frac{dk}{2\pi} e^{ixP+ikQ} \int dp dq e^{-ixp-ikq} H(p,q)$$

- Now we use this in the probability amplitude:

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}$$

The Path Integral

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}$$

- As we let δt → 0, we get an infinite number of integrals! This is a path integral – we integrate over every possible path between the two points. The result is:

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \, \mathcal{D}p \; exp\left[i \int_{t'}^{t''} dt \, (p(t)\dot{q}(t) - H(p(t), q(t))) \right]$$

where $\dot{q}$ is defined in the usual way

- Question: how do we integrate over a path?
  - We break the path parameter (q for example) into an infinite number of tiny points, then integrate over all tiny points. Hopefully a pattern emerges, and we don't have to do an infinite number of integrals! If not, can do so numerically, with a finite path size. See problem 6.1 (b).

Special Case

- Now let's assume that H is:
  - No more than quadratic in momentum
  - The quadratic term in momentum is independent of position
- In this case, this simplifies a lot:

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \; exp \left[i \int_{t'}^{t''} dt \; L\left(\dot{q}(t), q(t)\right) \right]$$

- By the way, we get L by:
  - Finding the stationary point (optimization) of the p-integral.
  - Plug this solution into $p\dot{q} - H$ to get L

Observation

- We've met our goal! Now what good did that do us? We can make some simple observations:

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \langle q''|e^{-iH(t''-t_1)}Qe^{-iH(t_1-t')}|q'\rangle$$

- We'll split this up into an infinite number of instantaneous eigenstates, everything is the same as before except for a new term of $\langle q_1|Q|q_j\rangle$

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \int \mathcal{D}p\ \mathcal{D}q\ q(t_1)e^{iS}$$

Observation 2

- Now we note that the following:

$$\int \mathcal{D}p\, \mathcal{D}q\ q(t_1)q(t_2)e^{iS} = \langle q'', t''|TQ(t_1)Q(t_2)|q', t'\rangle$$

- These operators have to be time-ordered, otherwise we can't split up the time interval as required.
  - This looks similar to the LSZ Formula

Functional Derivatives

- We need functional derivatives to proceed. These are defined by:

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2)$$

- Beyond this, you can "follow your nose" – all the other rules for derivatives still hold.

Observation 3

- We can add external forces like this:

$$H(p,q) \rightarrow H(p,q) - f(t)q(t) - h(t)p(t)$$

- Hence:

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}p\mathcal{D}q \; e^{i\int dt[p\dot{q} - H + fq + hp]}$$

- Now do three things:
  - take a bunch of functional derivatives, which will bring down a bunch of q's or p's.
  - Set f(t) = h(t) = 0, returning to the original formula (but now with a bunch of extra q's and p's.
  - Use observation 2 to turn the q's and p's into time-ordered operators.
- The result is:

$$\langle q'', t'' | TQ(t_1) \ldots P(t_n) \ldots | q', t' \rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)} \ldots \frac{1}{i}\frac{\delta}{\delta h(t_n)} \ldots \langle q'', t'' | q', t' \rangle \Big|_{f=h=0}$$

Other States

- Up to now we've only considered position eigenstates. What about other eigenstates?
  - Add these by adding in two complete sets of bases (here we'll use the ground state):

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty} \lim_{t''\to\infty} \int dq'' dq' \langle 0|q'', t''\rangle \langle q'', t''|q', t'\rangle \langle q', t'|0\rangle$$

  - We can write this as:

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty} \lim_{t''\to\infty} \int dq'' dq' \psi_0^*(q'') \langle q'', t''|q', t'\rangle_{f,h} \psi_0(q')$$

Special Case: The Ground State

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty}\lim_{t''\to\infty}\int dq''dq'\psi_0^*(q'')\langle q'',t''|q',t'\rangle_{f,h}\psi_0(q')$$

- This is a bit cumbersome. Let's let $|n\rangle$ be an eigenstate of the Hamiltonian. Now we have:

$$|q',t'\rangle = e^{iHt'}|q'\rangle$$

$$|q',t'\rangle = \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle$$

$$|q',t'\rangle = \sum_{n=0}^{\infty} \psi_n^*(q')e^{iE_nt'}|n\rangle$$

The Trick

- We replace H with (1-iε)H:
  - This gives a new term:

$$|q', t'\rangle = \sum_{n=0}^{\infty} e^{iHt'} e^{\varepsilon Ht'} \psi_n^*(q')|n\rangle$$

$$|q', t'\rangle = \sum_{n=0}^{\infty} \psi_n^*(q') e^{iE_n t'} e^{\varepsilon E_n t'} |n\rangle$$

  - As t' → -∞, all terms except the ground state are killed. So:

$$\lim_{t' \to -\infty} |q', t'\rangle = \psi_0^*(q')|0\rangle$$

The Trick, continued

$$\lim_{t' \to -\infty} |q', t'\rangle = \psi_0^*(q')|0\rangle$$

- Now we multiply by a non-orthogonal function and integrate.

$$\lim_{t' \to -\infty} \int dq' \chi(q') |q', t'\rangle = \int dq' \chi(q') \psi_0^*(q')|0\rangle$$

- On the right, this is a constant:

$$\lim_{t' \to -\infty} \int dq' \chi(q') |q', t'\rangle = k|0\rangle$$

- And the constant doesn't matter since it will be accounted for when the path integral is normalized.

The Trick, summary

- What did all this do for us?
  - If you use (1-iε)H rather than H, in the ket, then the initial state will be the ground state, for any reasonable boundary conditions.
  - This also holds for the bra and the final state.
  - So, any reasonable boundary condition will result in the ground state with this substitution.

The Trick: result

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty}\lim_{t''\to\infty}\int dq''dq'\psi_0^*(q'')\langle q'',t''|q',t'\rangle_{f,h}\psi_0(q')$$

- We'll use this formula.
  - Using the trick, we let H → (1-iε)H.
  - In exchange, we get to let q', q'' be whatever we want – let's choose the function that causes the integrated norm of ψ_0 to be one.

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q\ exp\left[i\int_{-\infty}^{\infty} dt\,(p\dot{q} - (1-i\varepsilon)H + fq + hp)\right]$$

Srednicki ends by rewriting this again, see 6.22 and 6.23.