QFT

Chapter 57: The path integral for photons

Chapter 8, Revisited

- Remember in chapter 8, we took on the scalar path integral directly?
  - We did the following:
    - Fourier transform
    - Shifted path integration variable
    - Did the path integral, using the “key point” form chapter 7– notice that this is consistent with the case f = 0, meaning that there is no external force, meaning that the system in the ground state will remain in the ground state with probability 1. Thus, the path integral vanishes.
  - We found that the path integral was:

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4x'\; J(x)\Delta(x-x')J(x')\right]$$

  - Then we took some functional derivatives to derive Wick’s Theorem

- Now we want to do the same thing for photons.
  - This is essentially trivial, but one key subtlety….

The Subtlety

- The problem is with step 2, shifting the integration variable.
  - In chapter 8, we had to invert $(k^2 + m^2)$ to get this to work out cleanly
  - Now, we have to invert $k^2 g^{\mu\nu} - k^\mu k^\nu$
  - But, the matrix cannot be inverted!

- Why not?
  - We can write $k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}$, where $P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2$
    - But P is a projection operator: $P^2 = P$.
    - Of course, projection operators have eigenvalues of 0 or 1 only.
      - $P^{\mu\nu} k_\nu = 0$ (by definition), so one eigenvalue is zero.
      - Recall the trace of the matrix is the sum of the eigenvalues. By definition, the trace of the matrix is 3. So, the other three eigenvalues are one.
    - But, if one eigenvalue is zero, the matrix is not invertible.

The Subtlety, continued

- Since we can't complete the square, we'll have to deal with this action directly:

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\tilde{A}_\mu(k)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\tilde{A}_\nu(-k) + \tilde{J}^\mu(k)\tilde{A}_\mu(-k) + \tilde{J}^\mu(-k)\tilde{A}_\mu(k)\right]$$

  - We decompose the field against components; the component that lies along k^μ does not contribute to the quadratic term, since, as already discussed, $P^{\mu\nu}k_\nu = 0$. It does not contribute to the linear terms either, by conservation of current.
  - We can now drop this component from the integral, and redefinine the path integral to be over onl ythe three basis vectors.
    - This is equivalent to imposing Lorentz Gauge.
  - Within the subspace orthogonal to k^μ, the projection operator is equivalent to the identity operator. So, within that subspace, it is easy to invert our matrix
    - The inverse of $k^2P^{\mu\nu}$ is $(1/k^2)P^{\mu\nu}$
  - Now we use our usual trick, replacing $k^2 - i\varepsilon$ to pick out the boundary conditions.

Conclusions

- Now that we’ve resolved this subtlety, we can continue the procedure of chapter 8.
  - We end up with our familiar expressions for the path integral and the photon propagator, this time in Lorentz/Landau gauge