QFT

Chapter 53: Functional Determinants

Overview

- Recall in chapter 44, we introduced Grassman numbers to derive equation 44.40, a general mathematical formula:

$$\int d^n\chi d^n\overline{\chi}\, \exp(\chi^\dagger M\chi + \eta^\dagger\chi + \chi^\dagger\eta) = (\det M)\exp(-\eta^\dagger M^{-1}\eta)$$

  - The fermionic path integral was just a special case of this.
  - In this special case, the determinant was independent of the fields and sources, and could just be absorbed into the overall normalization.
  - Here, we will ponder the meaning of this – will be needed later on when we get to non-Abelian Gauge theories.

Analysis #1

- We have from chapter 44 that

$$\int d^n z d^n \overline{z} e^{-i\overline{z}_i M_{ij} z_j} \propto (\det M)^{-1}$$

- In the case at hand, we have:

$$M = [-\partial_x^2 + m^2 - g\phi]\delta^4(x-y)$$

- But how do we take the determinant of this? First, we define:

$$M = \int d^4 y M_0(x,y)\widetilde{M}(y,z)$$

  - This factorization is non-intuitive, but Srednicki "guesses" the proper factorization and it is easy to verify that he is right
  - The advantage is that M_0 is defined to be independent of the background field φ, and can therefore be absorbed into the normalization

Analysis #1, continued

- This gives us:

$$Z(\phi) = (\det \widetilde{M})^{-1}$$

- As it turns out, the form needed to factor was:

$$\widetilde{M}(y, z) = \delta^4(y - z) - g\Delta(y - z)\phi(z)$$

- So now we need to take the determinant of this. We can use the general matrix relation

$$\det A = \exp \mathrm{Tr} \log A$$

- and it follows that we get:

$$Z(\phi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left[g^n \int d^4x_1 \ldots d^4x_n \Delta(x_1 - x_2)\phi(x_2) \ldots \Delta(x_n - x_1)\phi(x_1) \right]^n$$

Analysis #2

- Now we can redo this, treating the interaction term in the Lagrangian as an interaction (and phi as part of the vertex factor, rather than a separate field).
  - The only diagrams we can draw are circles of vertices.
  - Summing the diagrams, we rederive our result from analysis #1

Again

- Now we repeat both these analyses for fermions.
    - The difference is the negative sign we found in problem 51.1

- It's important to note that not only did we derive the negative sign, but we also showed that we can deal directly with these functional determinants.