QFT

Unit 5: The LSZ Reduction Formula

Overview

- If we start off with situation a, what are the odds that we end up with situation b? This (the S-matrix) can be calculated by the LSZ Reduction Formula, which we derive here.
- Our answer will be in terms of correlation functions, which we'll learn how to evaluate later.
- We'll also see for the first time:
  - Renormalization
  - States with more than one particle

Multi-Particle States

- A well-localized particle is given by:

$$a_1^\dagger = \int d^3k \; e^{\frac{-(\mathbf{k}-\mathbf{k_1})^2}{4\sigma^2}} a^\dagger(\mathbf{k})$$

- Time-evolving this, it spreads out; localized far from origin at t = ±∞.

- Now we wish to have an initial state with more than one particle. The following is an ingenious guess:

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t)|0\rangle$$

  - The interactions and time-dependence might cause problems, but we avoid that by working at t = ±∞ when it is very spread out.

The (unrefined) LSZ Formula

- Putting our results together, we have:

$$\langle f|i\rangle = \langle 0|a_{2'}(\infty)a_{1'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|0\rangle$$

- If we normalize both states properly (such that $\langle i|i\rangle = 1$ etc., then this shows the odds that two particles come in, interact, then leave.
  - Not necessarily one since particles could be created or destroyed.
  - Could add more operators to initial/final states to calculate probability of other outcomes.

The (refined) LSZ Formula

- Next we simplify. The result is:

$$\langle f|i\rangle = i^{n+n'}\int d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)\ldots d^4x' e^{-ik_1'x_1'}(-\partial_{1'}^2+m^2)\ldots\langle 0|T\phi(x_1)\ldots\phi(x_1')\ldots|0\rangle$$

(I won't go through the math, since something similar is done in problem 5.1)

- Objection: won't the $(-\partial^2+m^2)$ term vanish?
- Answer:
  - Only in the free theory. In an interacting theory, this (acting on φ), will equal the interacting terms.
  - Further, there is no φ (yet) for these operators to act on (though perhaps there will be once the correlation function is worked out).

Check our Assumptions!

- We’re done!
  - We don’t yet know what to do with the bra-ket at the end (the correlation function), but that’s a subject for another day.
- But, did we make any suspect assumptions?
  - The “hidden” math is fine, see problem 5.1.
  - The potential problem is our supposition that the creation operators for an interacting theory work the same as the creation operators for an free theory.

Bound States

- To look at this problem, we note that an interacting theory would have bound states:
  - Ground state with no particles (assumed to be unique).
  - Excited state with one particle, $E^2 = m^2 + k^2$
  - Excited state with n particles, $E2 \geq nm^2 + k_1^2 + \ldots + k_n^2$; can be anything due to relative momentum.
  - We disallow bound states with lower energy than the free states (at least for the next few chapters), since they are complicated.

- We can use this to check whether the interacting theory gives the same behavior (with respect to the creation/ annihilation operators) as the free theory

Renormalization: Ground State

- Consider:

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{-iPx}\phi(0)e^{iPx}|0\rangle$$

- Time-evolving the vacuum doesn't do much:

$$\langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle$$

- We don't know what this is – all we know is that it's a Lorentz-Invariant number.
- But we want it to be zero. If it's not zero, then $a_1^\dagger$ creates some linear combination of the ground state, which is bad.
  - This is because the mode expansion of φ now is analogous to that of the free field, with a → a_1.
- To enforce this condition, we'll just make the substitution:

$$\phi(x) \longrightarrow \phi(x) + v$$

Renormalization: 1-Particle States

- Now consider the same thing for a 1-particle state:
$$\langle p|\phi(x)|0\rangle = \langle p|e^{-iPx}\phi(0)e^{iPx}|0\rangle$$
- Time-evolving the bras and kets gives:
$$\langle p|\phi(x)|0\rangle = e^{-ipx}\langle p|\phi(0)|0\rangle$$
- This is a Lorentz-invariant function of p. The only Lorentz-invariant function of p is p^2, which is constant ($-m^2$). Hence, this is a constant.
- We want this to be one, just like in free-theory.
- To enforce this condition, we'll re-normalize φ.

Renormalization: 2-Particle States

- Now consider the same thing for a multi-particle state:

$$\langle p, n|\phi(x)|0\rangle = e^{-ipx}\langle p, n|\phi(0)|0\rangle$$

- The math here is complicated (see Srednicki 40-41), but the key points are that:
  - We want this to go to zero, since we don't want $a_1^\dagger$ to create multi-particle states from the vacuum.
  - This goes to zero of its own accord, so no further action is required.
    - The Riemann-Lebesgue Lemma is helpful here: as sine waves oscillate more and more rapidly, their integral gets closer and closer to zero.

Summary

- The LSZ Formula is:

$$\langle f|i\rangle = i^{n+n'}\int d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)\ldots d^4x' e^{-ik_1'x_1'}(-\partial_{1'}^2+m^2)\ldots\langle 0|T\phi(x_1)\ldots\phi(x_1')\ldots|0\rangle$$

- The LSZ Formula is valid provided that the $a_1^\dagger$ operators work similarly to the $a^\dagger$ operators.

$$\langle p,n|\phi(x)|0\rangle = e^{-ipx}\langle p,n|\phi(0)|0\rangle$$

$$\langle p|\phi(x)|0\rangle = e^{-ipx}\langle p|\phi(0)|0\rangle$$

Example

- Say that our Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{6}g\phi^3$$

- After shifting and rescaling, we have:

$$\mathcal{L} = -\frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + \frac{1}{6}Z_g g\phi^3 + Y\phi$$

- These four constants are fixed by:
  - The LSZ conditions (2)
  - $Z_m = 1$ so that the true mass is enforced
  - $Z_g = 1$ so that the true "g" (as measured in cross-sections) is enforced

What’s next?

- Next four sections are about these correlation functions.
  - Turns out there’s some interesting physics in here, including proto-Feynman diagrams
- Then, we’ll use LSZ to calculate some scattering amplitudes, cross-sections, etc.