QFT

Chapter 44: Formal Development of Fermionic Path Integrals

Grassmann Numbers

- Grassmann numbers are numbers that anti-commute by definition.
- Note that $\psi^2 = 0$ if ψ is an anti-commuting Grassmann number
- A function of such a variable $f(\psi)$ can be written as a Taylor Expansion; due to the property above, the even terms cancel and the odd terms can be reduced to first-order. Thus:

$$f(\psi) = a + \psi b$$

Grassmann Numbers, again

- Do we want the function itself to be commuting or anticommuting? We can define it either way.
  - In general, we'll be interested in commuting functions.
  - We'll therefore need a to commute and b to anti-commute. Thus,

$$f(\psi) = a + \psi b = a - b\psi$$

- How do we take the derivative of this?
  - Define left-derivative if ψ is on the left. Then differentiate normally
    - This is what we usually mean, unless otherwise stated, or as below.
  - Define right-derivative if ψ is on the right. Then differentiate normally
    - This is what we usually mean when we're dealing with a canonical momentum for a fermionic field.

Grassmann Numbers, again

- We also need an Grassmann integral.
  - This should be linear and invariant under shifts of the dependent variable
  - With the Grassman expansion discussed before, the only way to do this is to have a constant (which can be ordinary or Grassmann)

- Finally, we define complex Grassmann numbers:
  - $\chi = \psi_1 + i\psi_2$

- Integrals like this are evaluated as before:

$$\int d\chi \; d\overline{\chi} \; f(\chi, \overline{\chi}) = d$$

- For example

$$\int d\chi \; d\overline{\chi} \; e^{m\overline{\chi}\chi} = m$$

Conclusion

- At this point all the new stuff is introduced, it's just a matter of going through a very specific derivation to get to the following general mathematical theorem:
  - As usual, no point going through the math again, it's in the book.

$$\int d^n\chi \, d^n\overline{\chi} \;\exp(\chi^\dagger M\chi + \eta^\dagger\chi + \chi^\dagger\eta) = (\det M)\exp(-\eta^\dagger M^{-1}\eta)$$

- It is clear that our result from last chapter is just a special case of this.