QFT

Chapter 43: The path integral for fermion fields

Review

- If only we could solve path integrals directly!
  - Path integral would tell us transition amplitude
  - Techniques of chapter 11 to convert transition amplitude to decay rate or cross-section
  - And that's it! Decay rates and cross-sections are just about the whole point of QFT
    - The "theoretical" underpinnings like symmetries and renormalization would still come up since we still have to deal with the Lagrangian
    - But, stuff like 1-loop corrections, Feynman diagrams, and breakdown of the perturbative expansion would no longer be concerns, since we would no longer be using perturbation theory.

- But, the math is too hard.
  - Instead, we figure out the free-field path integral and solve it.
    - Next two chapters are about this
  - Then, the interacting-field path integral can be written in terms of the free-field one, and solved through perturbation theory

Overview

- The next two chapters are devoted to the path integral for free Fermion fields:
  - Here we follow our nose(s) to arrive at the proper form
  - In the next chapter we prove it a little more rigorously

- After that, we add an interaction term, which allows us to draw Feynman diagrams, compute the correlation functions, plug into the LSZ formula, and calculate cross-sections. One new subtlety that will arise is the spins.

- After that, we compute beta functions and deal with one more subtlety (functional determinants).

- And that's it for spin-1/2.

Result from Scalar Fields

- The path integral for scalar fields is:

$$Z(J) = exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]$$

- where:

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}$$

is the scalar propagator, the thing that, when acted on by the Klein-Gordon wave operator, gives the delta function.

- Further, recall that we evaluate correlation functions by:

$$\langle 0|T\phi(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\ldots Z_0(J)\Big|_{J=0}$$

Complex Fields, Functional Derivatives of Dirac Fields

- We treat the complex conjugate of the field as a totally separate field:

$$\langle 0|T\phi(x_1)\ldots\phi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J^\dagger(x_1)}\ldots\frac{1}{i}\frac{\delta}{\delta J(y_1)}\ldots Z_0(J^\dagger, J)\bigg|_{J=J^\dagger=0}$$

- For the Dirac field, we'll use η rather than J for the sources.

- Note that we have:

$$\frac{\delta}{\delta\eta(x)}\int d^4y\left[\overline{\eta}(y)\Psi(y)+\overline{\Psi}(y)\eta(y)\right] = -\overline{\Psi}(x)$$

where the minus sign is due to the anti-commutation

- a functional derivative with respect to an anti-commuting function is itself defined to be anti-commutating

Dirac Path Integrals

- In analogy with the complex case, we can guess the appropriate forms for the Dirac Path Integral:

$$\langle 0|T\Psi(x_1)\ldots\Psi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x_1)}\ldots i\frac{\delta}{\delta\eta(y_1)}\ldots Z_0(\overline{\eta},\eta)\Big|_{\eta=\overline{\eta}=0}$$

$$\begin{aligned} Z_0(\eta,\overline{\eta}) &= \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\exp\left[i\int d^4x(\mathcal{L}_0+\overline{\eta}\Psi+\overline{\Psi}\eta\right] \\ &= \exp\left[i\int d^4x d^4y\overline{\eta}(x)S(x-y)\eta(y)\right] \end{aligned}$$

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{(-\not{p}+m)e^{ip(x-y)}}{p^2+m^2-i\epsilon}$$

$$(-i\not{\partial}_x+m)S(x-y) = \delta^4(x-y)$$

Interactions

- As before, we will write:

$$Z(\overline{\eta}, \eta) \propto \exp\left[i \int d^4x \; \mathcal{L}_1 \left(i \frac{\delta}{\delta\eta(x)}, \frac{1}{i} \frac{\delta}{\delta\overline{\eta}(x)} \right)\right] Z_0(\overline{\eta}, \eta)$$

where L_1 is the interacting part of the Lagrangian.
  - The overall normalization is fixed by Z(0,0) = 1
  - As before, we'll expand both terms and draw Feynman diagrams

- Two extra complications:
  - Spinor indices
  - Extra minus signs from anticommutation

Majorana Fields

- In analogy with the Dirac case, we have

$$\langle 0|T\Psi_a(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta\eta_a(x_1)}\ldots Z_0(\eta)\bigg|_{\eta=0}$$

$$\begin{aligned} Z_0(\eta) &= \int \mathcal{D}\Psi \exp\left[i\int d^4x(\mathcal{L}_0 + \eta^T\Psi)\right] \\ &= \exp\left[-\frac{i}{2}\int d^4x d^4y\ \eta^T(x)S(x-y)\mathcal{C}^{-1}\eta(y)\right] \end{aligned}$$

where this time the Feynman Propagator has an extra term of C^{-1}; this is the inverse of the Majorana wave operator.

Note

- Everything here was just a guess by analogy, though it would be easy enough to reproduce the results of section 42.

- Instead, we'll turn to a more formal derivation in the next chapter.