QFT

Unit 4: The Spin-Statistics Theorem

The Spin-Statistics Theorem

- Anti-commutation relations (fermions) may be used only on half-integer spin particles.
- Commutation relations (bosons) may be used only on integer spin particles.
- We saw a hint of this last time, because we got a trivial L from spin-0 anti-commutators.
  - But not yet clear why this is the case, or if the problem can be fixed.
- The rest of this section is to prove the theorem for spin = 0.

Outline

1. Prove that φ^+ and φ^- yield local, Lorentz-Invariant interactions.
2. Require the transition amplitude to be constant in different frames.
3. Show that this requirement is impossible (for spin 0) for anti-commutators, and for any field operator other than φ.

Local, Lorentz-Invariant Interactions

- Let's consider the Hamiltonian for free, non-interacting spin-0 fields, with $\varepsilon_0 = \Omega_0$.

$$H_0 = \int \widetilde{dk} \omega a^\dagger(\vec{k}) a(\vec{k})$$

- Now let's define φ^+, φ^-

$$\phi^+(x,0) = \int \widetilde{dk} e^{ik\cdot x} a(k)$$

$$\phi^-(x,0) = \int \widetilde{dk} e^{-ik\cdot x} a^\dagger(k)$$

Local, Lorentz-Invariant Interactions

- Time-evolving these:

$$\phi^+(x,t) = \int \widetilde{dk} e^{ikx} a(k)$$

$$\phi^-(x,t) = \int \widetilde{dk} e^{-ikx} a^\dagger(k)$$

- Notice two things:
  - The sum of these is just φ
  - φ^+ and φ^- are Lorentz scalars, ie when sandwiched between the unitary Lorentz matrices, their argument is Lorentz-transformed.

Schrodinger & Heisenberg Pictures

- When systems are time-evolved, there are two ways of thinking about it:
  - Heisenberg: The initial and final states are constant, but the Hamiltonian in the evolution operator is time-dependent.
  - Schrodinger: The operator is constant, but the states evolve in time.

Transition Amplitudes

- Let's choose the Heisenberg picture. The transition amplitude is:

$$\tau_{f\leftarrow i} = \langle f|Texp\left[-i\int_{-\infty}^{\infty} dtH_I(t)\right]|i\rangle$$

- Let's chop this integral up into many little pieces. How can we interpret the time-ordering symbol in light of relativity?
  - Time-ordering must be frame independent.

Transition Amplitudes

- If separation is time-like, no problem. If separation is space-like, we must require:

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0$$

- But in fact:

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m}{4\pi r^2} K_1(mr)$$

(see problem 4.1)

This is never zero.

Resolving this Problem

- There must be some linear combination of φ^+ and φ^- that will work – otherwise we can't add even a simple interaction term to the Lagrangian.

- Let's try the most general linear combination. We find:

$$[\phi_\lambda(x), \phi_\lambda(x')]_\mp = \lambda(1 \mp 1)\frac{m}{4\pi^2 r}K_1(mr)$$

$$[\phi_\lambda(x), \phi_\lambda^\dagger(x')]_\mp = (1 \mp |\lambda|^2)\frac{m}{4\pi^2 r}K_1(mr)$$

Resolving the Problem, cntd.

- This is zero if and only if:
  - Choose anti-commutators
  - Choose $|\lambda| = 1$

- But this second requirement gives us (up to a phase shift) φ again!

- We tried to start with a, $a^\dagger$, but it seems that the fundamental object is instead φ.

- Further, we must choose the anti-commutation operators to avoid a trivial L.