Status

• It may seem like we’re drifting. In fact, we’re making progress.
  • QFT is based on the assumption of Lorentz-Invariance
  • In chapter 33, we decided that the Lorentz Group needs to be put in the (1,2) or (2,1) representation if we are to consider spin $\frac{1}{2}$ particles.
  • In chapter 34, we wrote the Lorentz Group in the spinor representation and worked out the generators
  • In chapter 35, we worked out invariant symbols in the spinor representation, and got a nice notation
  • In chapter 36, we figured out the Lagrangian for our theory, and also the fields $\Psi$ – these are analogous to $\phi$ from part 1.
Status

• How does our new theory for spin-1/2 particles compare to our old scalar theory?

• We already have our free-field Lagrangian, our field $\Psi$ and our equation of motion (the Dirac Equation). We’ve also figured out some of the rules for mathematically dealing with spinors.

• But, we don’t have an LSZ formula or Feynman rules. We don’t know how to evaluate correlation functions, nor do we have commutation relations.

• Our next task is to rebuild this stuff, so that we can compute cross-sections and decay rates for spin-1/2 particles. In this chapter:
  • We figure out the (anti)commutation relations for the fields
  • We solve the Dirac Equation to get a mode expansion for the field
Anticommutation

• The spin-statistics theorem tells us that Weyl fields anticommute.
  • We will show in problem 37.1 that this extends to the fermion fields $\psi$ and $\bar{\psi}$.

• Explicitly:

\[
\{ \Psi_\alpha(x, t), \Psi_\beta(y, t) \} = 0
\]

\[
\{ \Psi_\alpha(x, t), \bar{\Psi}_\beta(y, t) \} = (\gamma^0)_{\alpha\beta} \delta^3(x - y)
\]
Slash Notation

• We introduce the Feynman Slash, which is:

\[ \not{a} = a_\mu \gamma^\mu \]

• The Dirac equation can therefore be written:

\[ ( -i \not{\partial} + m ) \Psi = 0 \]
Solving the Dirac Equation

• Note that we can square the Dirac Equation and recover the Klein-Gordon equation. So, $\Psi$ obeys the Klein-Gordon equation, and must have plane-wave solutions:

$$\Psi(x) = u(\vec{p})e^{ipx} + v(\vec{p})e^{-ipx}$$

• Plugging this solution into the Dirac equation, we find that we must also require

$$(\not{p} + m)u(\vec{p}) = 0$$

$$(\not{p} - m)v(\vec{p}) = 0$$

• In fact, there are two possible solutions for $u$ and for $v$. We’ll explore these in the next chapter.
Mode Expansion of $\psi$

- The most general solution of the Dirac Equation is therefore:

$$
\Psi = \sum_{s=\pm} \int \tilde{d}p \left[ b_s(\vec{p}) u_s(\vec{p}) e^{ipx} + d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ipx} \right]
$$

where $b$ and $d$ are some sort of operator. In fact, we’re about (chapter 39) to go to quantize the field, at which point this will become creation/annihilation operators for fields.

First, though, we investigate the properties of $u$ and $v$. 