QFT

Chapter 37: Canonical Quantization of Spinor Fields I

Status

- It may seem like we're drifting. In fact, we're making progress.
  - QFT is based on the assumption of Lorentz-Invariance
  - In chapter 33, we decided that the Lorentz Group needs to be put in the (1,2) or (2,1) representation if we are to consider spin ½ particles.
  - In chapter 34, we wrote the Lorentz Group in the spinor representation and worked out the generators
  - In chapter 35, we worked out invariant symbols in the spinor representation, and got a nice notation
  - In chapter 36, we figured out the Lagrangian for our theory, and also the fields Ψ – these are analogous to φ from part 1.

Status

- How does our new theory for spin-1/2 particles compare to our old scalar theory?

- We already have our free-field Lagrangian, our field Ψ and our equation of motion (the Dirac Equation). We've also figured out some of the rules for mathematically dealing with spinors.

- But, we don't have an LSZ formula or Feynman rules. We don't know how to evaluate correlation functions, nor do we have commutation relations.

- Our next task is to rebuild this stuff, so that we can compute cross-sections and decay rates for spin-1/2 particles. In this chapter:
  - We figure out the (anti)commutation relations for the fields
  - We solve the Dirac Equation to get a mode expansion for the field

Anticommutation

- The spin-statistics theorem tells us that Weyl fields anti-commute.
  - We will show in problem 37.1 that this extends to the fermion fields ψ and ψ bar.

- Explicitly:

$$\{\Psi_\alpha(x,t), \Psi_\beta(y,t)\} = 0$$

$$\{\Psi_\alpha(x,t), \overline{\Psi}_\beta(y,t)\} = (\gamma^0)_{\alpha\beta}\delta^3(x-y)$$

Slash Notation

- We introduce the Feynman Slash, which is:

$$\not{a} = a_\mu \gamma^\mu$$

- The Dirac equation can therefore be written:

$$(-i \not{\partial} + m)\Psi = 0$$

Solving the Dirac Equation

- Note that we can square the Dirac Equation and recover the Klein-Gordon equation. So, Ψ obeys the Klein-Gordon equation, and must have plane-wave solutions:

$$\Psi(x) = u(\vec{p})e^{ipx} + v(\vec{p})e^{-ipx}$$

- Plugging this solution into the Dirac equation, we find that we must also require

$$(\not{p} + m)u(\vec{p}) = 0$$
$$(-\not{p} + m)v(\vec{p}) = 0$$

- In fact, there are two possible solutions for u and for v. We'll explore these in the next chapter.

Mode Expansion of ψ

- The most general solution of the Dirac Equation is therefore:

$$\Psi = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\vec{p}) u_s(\vec{p}) e^{ipx} + d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ipx} \right]$$

where b and d are some sort of operator. In fact, we're about (chapter 39) to go to quantize the field, at which point this will become creation/annihilation operators for fields.

First, though, we investigate the properties of u and v.