QFT

Chapter 36: Lagrangians for Spinor Fields

Constructing Lagrangians

- We want a Lagrangian for a left-handed spinor field ψ
  - Must be Lorentz Invariant
  - Must be Hermitian
  - Must be quadratic in ψ and its Hermitian conjugate
    - (because we want free particles, ie plane waves)

- Recall also the invariant objects that we can use, depending on the number of indices we have:
  - ε^{ab}
  - $\sigma^{\mu ab}$

Free Lagrangian

- The Free Lagrangian turns out to be:

$$\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi - \frac{1}{2} m\psi\psi - \frac{1}{2} m\psi^\dagger \psi^\dagger$$

- Why?
  - What can we do with no derivatives? The only possible quadratic term is ψψ or $\psi^\dagger\psi^\dagger$. m is arbitrary but must have the dimensionality of mass. To make this Hermitian, we need both the second and third terms.
    - Why didn't we take the conjugate of m? Choosing the phases of both m and ψ is stupid, since setting one will affect the other. Instead, we define m to be real and positive in the first place, and allow ψ to decide the phase.
  - What about with derivatives?
    - Before we took derivatives on both fields. But that leaves something that's unbounded below.
    - Instead, let's take the derivative of only one. We'll need a factor of $\sigma^{\mu ab}$ to get the indices to work out, and a factor of i to help with Hermiticity
      - Srednicki explains that this is actually not Hermitian, but it is Hermitian up to a total divergence, and will vanish when we integrate to get the action. So close enough.

The Dirac Equation

- Recall that our scalar Lagrangian had the Klein-Gordon Equation as its equation of motion
  - This was a decision, then we cooked up the Lagrangian to account for it
- This time, the logic works in reverse: we cooked up the simplest Lagrangian that met our anticipated needs, and calculate the equation of motion
  - The result is the Dirac Equation!
- We first saw the Dirac Equation in chapter 1
  - We recognized then that it was Lorentz Invariant and even compatible with Quantum Mechanics, but we didn't know how to interpret the 4x4 matrices. We had expected the spin-1/2 object to have 2x2 matrices
  - Now we know – these "gamma matrices" are *operators* that can act on left-handed or right-handed spinors. We'll later interpret these spinors as particles and antiparticles. Each spinor has two components corresponding to spin, as expected..

The Dirac Equation, cntd.

- No point in deriving everything again, but let's state the key results.

- The Dirac equation is:

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0$$

- The gamma matrices are <u>defined</u> by:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

(any more explicit form, like eq. 36.7, is a particular representation)

- The left-handed and right-handed spinors are combined into the "*Majorana Field*," given by:

$$\Psi = \begin{pmatrix} \psi_c \\ \psi^{\dagger \dot{c}} \end{pmatrix}$$

  - You may know that Majorana particles are their own antiparticle. This definition suggests why.

Majorana vs. Dirac Fields

- What if we have *two* left-handed spinor fields?
  - The Lagrangian is of course:

$$\mathcal{L} = i\psi_a^\dagger \sigma^\mu \partial_\mu \psi_a - \frac{1}{2} m \psi_a \psi_a - \frac{1}{2} m \psi_a^\dagger \psi_a^\dagger + i\psi_b^\dagger \sigma^\mu \partial_m u \psi_b - \frac{1}{2} m \psi_b \psi_b - \frac{1}{2} m \psi_b^\dagger \psi_b^\dagger$$

  - This is invariant under "rotation" between the two fields, ie there is an SO(2) symmetry.

  - Defining χ and ζ to be the symmetric and antisymmetric combinations, respectively, we can rewrite the Lagrangian and recast the symmetry as a U(1) symmetry:

$$\mathcal{L} = i\chi^\dagger \sigma^\mu \partial_\mu \chi + i\xi^\dagger \sigma^\mu \partial_\mu \xi - m\xi\chi - m\xi^\dagger \chi^\dagger$$

  - This time, the equation of motion (derived as before) involves the *Dirac Field*

$$\Psi = \begin{pmatrix} \chi_c \\ \xi^{\dagger \dot{c}} \end{pmatrix}$$

Utility of the Dirac Field

- The Dirac field can completely replace the two spinors that it contains. In this way, the Lagrangian and Noether current can be calculated and stated in terms of the fields

- We won't go through the math here, but the results are:

$$\mathcal{L} = i\Psi\gamma^\mu\partial_\mu\Psi$$

$$j^\mu = \Psi\gamma^\mu\Psi$$

- Can we do the same for the Majorana field? Yes, but first we need to talk about charge conjugation…

Charge Conjugation

- Notice that our Lagrangian

$$\mathcal{L} = i\chi^\dagger \sigma^\mu \partial_\mu \chi + i\xi^\dagger \sigma^\mu \partial_\mu \xi - m\xi\chi - m\xi^\dagger \chi^\dagger$$

is invariant under exchange of ζ and χ.

- We can impose this symmetry by means of the charge conjugation operator.

$$C = \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

- Acting on the field, the charge conjugate operator just reverses the position of the spinors.

- Srednicki also derives other properties of the C operator.

Utility of the Majorana Field

- By following the same steps as for the Dirac Field, we obtain:

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \frac{1}{2}m\Psi\Psi$$

- but this is not sufficient! To solve for the equation of motion, we have to use the fact that

$$\Psi = \Psi^{C} = \mathcal{C}\overline{\Psi}^{T}$$

in fact, Srednicki directly rewrites the Lagrangian to incorporate this

γ_5

- We choose γ_5 (the 5 is just a conventional choice; it is not truly orthogonal to the other gamma matrices) to help define projection matrices, mapping us from Ψ back to the spinors

- This matrix can be expressed as the product of i and the other four γ matrices (in order).