QFT

Chapter 34: Left- and Right-Handed Spinors

Lorentz Transformations

- Recall from ch. 2 that Lorentz transformations in (1,1) are:

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

- In (2,1) we have to sum over both indices of our field operator, so we need to introduce a new matrix L, which is also in the (2,1) representation:

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b\psi_b(\Lambda^{-1}x)$$

- Recall that we require the Λs to satisfy the group composition rule; we also require that of the Ls.

- Further, just as the infinitesimal U goes as δ + M, the infinitesimal L goes as δ + S. S and M follow the same commutation relations.

Explicit form for S

- Remember that anything that follows the Lie Algebra could be S. The convention is to use Pauli Matrices:

$$(S_L^{ij})_a^{\ b} = \frac{1}{2}\varepsilon^{ijk}\sigma_k$$

$$(S_L^{k0})_a^{\ b} = \frac{i}{2}\sigma_k$$

- All this still works for the (1,2) representation, except the matrices must be (1,2) matrices. To keep them separate, we use dotted indices.
    - It turns out that the S matrices are negative conjugates of each other.

Lorentz Invariance of the Levi-Cevita Symbol

- Consider this Lorentz Transformation:

$$U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a^{\ c}(\Lambda)L_b^{\ d}(\Lambda)C_{cd}(\Lambda^{-1}x)$$

- Question: is it possible to reduce C to smaller groups that do not mix with each other under Lorentz Transformations?
  - C has two (2,1) indices. So, we're asking about (2,1) x (2,1). By the Clebsch-Gordon expansion, we know that this is $(1,1)_A$ x $(3,1)_S$.
  - This means that we should be able to decompose into a symmetric and antisymmetric part:

$$C_{ab}(x) = \varepsilon_{ab}D(x) + G_{ab}(x)$$

  - Does this work with the equation above? The scalar part gives the desired relation (same as in chapter 2), so it follows that the Levi-Cevita symbol must be invariant under a Lorentz-Transformation.

Levi-Cevita Symbol

- We define the Levi-Cevita symbol via:

$$\varepsilon^{12} = \varepsilon_{21} = +1$$
$$\varepsilon^{21} = \varepsilon_{12} = -1$$

- This will apply for both dotted and undotted indices.

- The 4-D symbol works the same way, where the normalization is

$$\varepsilon^{0123} = +1$$

Fields in (2, 2) Representation

- A field carrying two indices, one dotted and one undotted, is in the (2,2) representation.
- But 2,2 is the vector representation. How can we get to the "normal" vector representation, $A_\mu(x)$? Let's define another invariant symbol to do this:

$$A_{a\dot{a}}(x) = \sigma^{\mu}_{a\dot{a}} A_{\mu}(x)$$

- We'll demonstrate later that a good choice is:

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma})$$

Arbitrary Field Decomposition

- Remember in the previous section, we said we can decompose any field like this:

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}g^{\mu\nu}T(x)$$

- The two vector indices combine via:

$$(2,2)\otimes(2,2) = (1,1)_S \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S$$

  - T is a scalar, so it must be the (1,1)
  - S is a symmetric, traceless 4x4 matrix. It therefore has 9 independent components, and so has to be (3,3)
  - A is an antisymmetric 4x4 matrix. From the above, we expect it to be (1,3) + (3,1).
    - This is a spinor with dotted indices plus a spinor with undotted indices.
    - We should therefore be able to decompose A into a field G and its Hermitian Conjugate.

Conclusions

- We conclude that an arbitrary field can be decomposed to this irreducible representation:

$$B^{\mu\nu}(x) = G^{\mu\nu}(x) + G^{\dagger\mu\nu} + S^{\mu\nu}(x) + \frac{1}{4}g^{\mu\nu}T(x)$$

- Srednicki derives the relation of G and A in the text.

- Note that G is self-dual, meaning that

$$G^{\mu\nu}(x) = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}(x)$$