QFT

Chapter 33: Representations of the Lorentz Group

Lorentz Transformations Revisited

- In chapter two, we discussed Lorentz Transformations of the form:

$$\phi(\Lambda x) = U(\Lambda)\phi(x)U^{-1}(\Lambda)$$

- But why must this only work for scalars? Can't we Lorentz Transform vectors? Tensors?
  - Yes, we can transform vector fields (one index) or tensor fields (two indices). Each index transforms separately.
  - Further, note that:
    - Symmetric matrices transform into symmetric matrices
    - Antisymmetric matrices transform into antisymmetric matrices
    - The trace transforms as a scalar field.
  - Thus, we can decompose any tensor into (at least) three components, each of which transforms separately (no "mixing"):

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}g^{\mu\nu}T(x)$$

Irreducible Representation

- Can we decompose these fields any more? How do we know when we have irreducible representations?

- To answer this, we remember our results from chapter 2.
  - The M tensor represents the generator of the Lorentz Group
  - The commutation relations of the Ms specify the Lie Algebra of the group.
  - M is antisymmetric, so it has 6 independent components. These can be identified as (for example) the three Js and the three Ks, representing rotations and boosts.

N

- Let's rewrite our Js and Ks like this:

$$N_i = \frac{1}{2}(J_i - iK_i)$$
$$N_i^\dagger = \frac{1}{2}(J_i + iK_i)$$

- Thus:

$$[N_i, N_j] = i\varepsilon_{ijk}N_k$$
$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk}N_k^\dagger$$

- This looks exactly like the angular momentum operators!

Angular Momentum

- Recall this result from quantum mechanics:

$$[J_i, J_j] = i\ \varepsilon_{ijk}\ J_k$$

  - We can find three Hermitian matrices that satisfy this requirement
  - These three matrices are the inequivalent, irreducible representations of the Lie Algebra of SU(2)
  - This works in any dimensionality: we can find three irreducible 4x4 matrices (for example); the eigenvalues of these matrices will be (half)-integers from –j to j, where j=(N-1)/2 (3/2 in this case)

- On the previous slide, we showed that we have *two* relations like these for the Lorentz group.
  - A particular representation must then be specified with two *j*s, which we label as (2j+1, 2j'+1).
  - Number of components in a representation is (2j+1)(2j'+1). The others are deterministic.

What *is* a representation?

- In QFT, particles are represented by a field operator of a particular dimension
  - If I want to represent a spinless particle, I can use a function of x and t. This is a scalar.
  - If I want to represent a fermion, I need two functions of x and t: one for the spin-up component and one for the spin down component. This is a spinor.
  - Similarly, bosons need three functions of x and t. This is a vector.

- Also, particles must be invariant under the Lorentz Algebra, ie under:

$$[N_i, N_j] = i\varepsilon_{ijk}N_k \qquad\qquad [N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk}N_k^\dagger$$

- But what are the dimensionality of i and j (i, j)?
  - (1,1): scalar.
  - (2,1): left-handed spinor.
  - (1,2): right-handed spinor.
  - (2,2): vector

- The first three are obvious. Why is (2,2) identified as the vector?
  - Need four components. So we have (2,2), (4,1) or (1,4).
  - (1,4) and (4,1) have allowed angular momenta of 3/2 only. No good!
  - (2,2) has allowed angular momenta of 0 or 1. This is perfect: a singlet for the time-component, and a vector for the space components.