QFT

Unit 3: Canonical Quantization
(of Scalar Fields)

Overview

- We' ll develop our first relativistic quantum field theory.
  - Will be valid for a universe with:
    - Only spin-zero bosons, called φ.
    - No φ-φ interactions
    - The Klein-Gordon equation as the equation of motion
  - The Hamiltonian for our theory will be:

$$H = \int \widetilde{dk} \omega a^\dagger(k) a(k) + (\mathcal{E}_0 - \Omega_0) V$$

Developing Intuition

- Recall our result from chapter one

$$H = \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) + \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})$$

- Here, we're only concerned with free, non-interacting particles, so this simplifies drastically:

$$H = \int d^3x \; a^\dagger(x) \left(-\frac{\hbar^2}{2m}\nabla^2 \right) a(x)$$

Developing Intuition

- Now we want to simplify this further:
  - Units: we set $\hbar = c = 1$ and can restore these units by dimensional analysis if necessary
  - Fourier Transforms: we define

$$\tilde{a}(p) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-ip\cdot x} a(x)$$

    where a(p) follows the same commutation relations as a(x). We can think of $a^\dagger(p)$ as creating a particle with momentum p (and indeterminate position, by Heisenberg)
  - The result is:

$$H = \int d^3p \frac{1}{2m} p^2 \tilde{a}^\dagger(p)\tilde{a}(p)$$

Developing Intuition

$$H = \int d^3p \frac{1}{2m} p^2 \tilde{a}^\dagger(p)\tilde{a}(p)$$

- If this acts on a single particle state, the energy eigenvalue will be $p^2/2m$.
- To make this relativistic, we'll use Einstein's formula instead. In our new units, H becomes:

$$H = \int d^3p (\mathbf{p}^2 + m^2)^{1/2} \tilde{a}^\dagger(\mathbf{p})\tilde{a}(\mathbf{p})$$

- This is intuitive, but non-rigorous. Did it work out anyway?
  - Only for bosons. You'd expect this argument to work out for fermions as well, but that case fails when done rigorously.
    - This makes sense – what the hell is a spin-0 fermion? More on this next time.
  - Only when the equation of motion is the Klein-Gordon Equation (not the Schrödinger Equation, as one would expect from this "proof")
  - We'll spend the rest of the chapter redoing this result rigorously.

Re-Derivation: Starting Point

- We start with classical real field φ(x)
  - φ(x) is like temperature: returns real value for each point in space
  - Totally classical: no factors of hbar!

- We want this to be Lorentz Invariant from the beginning. If only we had an equation that were Lorentz Invariant…
  - Klein-Gordon Equation! Choose this as equation of motion.

$$(\partial^2 + m^2)\phi(x) = 0$$

  - Remember: totally classical! No factors of hbar. To get the dimensions to work out, m is not a mass, but a constant with dimensions of length^{-1}.
  - Objection: The Klein-Gordon Equation is wrong!

    Response: No, it's just not consistent with QM. But for (relativistic) elementary particles, QM is "wrong," so we don't care if we're consistent with QM or not.
    - Better objection is that it is *arbitrary* – I have as yet given no indication that the equation of motion couldn't be some other Lorentz-Invariant equation. But this is our axiom, and it's justified by the experimental success of QFT.
      - See prbm. 3.4, which derives it assuming only the spacetime translation operator.

Lagrangian

- This equation of motion results from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0$$

(Srednicki sketches proof; see problem 3.5(a) for similar proof with more detail)

Solution of Klein-Gordon Equation

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{f(k)} \left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]$$

$$w = +\sqrt{k^2+m^2}$$

- We must impose "real-ness" directly:

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{f(k)} \left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]$$

- Then rewrite in 4-vector notation

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{f(k)} \left[a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right]$$

Lorentz-Invariant Differential

- A Lorentz-Invariant Differential would be nice.
  - Lorentz-Invariant condition (for real particles) is $k^2 = -m^2$.
  - Hence, $d^4k\delta(k^2 + m^2)\theta(k^0)$ is Lorentz-Invariant
    - Require on-shell
    - Disallow negative-energy solutions
- Now do 0-dimensional integral.
  - Result is $d^3k/2\omega$.
- For convenience, we'll normalize differently, choose Lorentz-Invariant differential to be:

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3 2\omega}$$

Solution is now:

$$\phi(\mathbf{x},t) = \int \widetilde{dk}\,\left[a(\mathbf{k})e^{ikx} + a^*(k)e^{-ikx}\right]$$

- Next step is to use this explicit form in the Lagrangian and remember that $H = \Pi\dot{\phi} - \mathcal{L}$ The result is:

$$H = -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\omega\left(a^*(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^*(\mathbf{k})\right)$$

(proved in Srednicki. I won't go through the proof myself, since problem 3.5e is very similar)

Quantize the Field

- This is the key step. Promote q (for us, φ) and p (for us, Π) to operators with commutation relations.
  - From this, derive commutators for a and $a^\dagger$.
  - Use this to rewrite the Hamiltonian.

$$H = \int \widetilde{dk} \omega a^\dagger(k) a(k) + (\mathcal{E}_0 - \Omega_0) V$$

  - Same result as our "intuitive" argument (just different notation)

Ultraviolet Cutoff

- Recall that our solution is:

$$H = \int \widetilde{dk} \omega a^\dagger(k) a(k) + (\mathcal{E}_0 - \Omega_0)V$$

- For the moment, Ω_0 is arbitrary, so we'll set it equal to ε_0.
- But ε_0 is defined by $\mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3} \int d^3k\ \omega$
  - If we don't want this to be infinite, put an upper limit (ultraviolet cutoff) on the integral.
  - Physically justified if QFT breaks down at given energy

What about Fermions?

- We assumed commutation relations, so we have bosons.
- Now let's assume anti-commutation relations.

$$H = -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\omega\left(a^*(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^*(\mathbf{k})\right)$$

will become.

$$H = -\Omega_0 V$$

Something's wrong! For the moment, have to ignore "spin-0 fermions"

Realistic?

- Are there any spin-0 (scalar) bosons?
  - Yes: pion and Higgs Boson, for example

- Does this theory describe those particles accurately?
  - Yes, provided that they don't interact with each other (or anything else) and are free.

What's Next?

- We're done – we have a relativistic QFT. But, it sucks!
  - A bunch of scalar bosons that don't interact with each other or any other particles or fields is not very interesting.
- Next time, we'll resolve this question about why spin-0 fermions are no good.
- After that, we'll develop tools needed for an interacting quantum field theory.

4-D Fourier Transformation

- In the problems (and later on), you'll need the 4-D Fourier Transforms (Srednicki 8.6):

$$\overline{\phi}(k) = \int d^4x e^{-ikx} \phi(x)$$

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \overline{\phi}(k)$$