# QFT

## Chapter 26: Infrared Divergences

# The Problem

- In section 20, we computed the $\varphi\varphi \to \varphi\varphi$ scattering amplitude in $\varphi^3$ theory in six dimensions in the high energy limit.

- The problem is that this includes a term proportional to $\ln(s/m^2)$, which blows up in the limit $m \to 0$.

- This means that we have made a mistake – in fact, we've made two mistakes. Both involve the fact that for a massless (or nearly massless) particle, there is a no (or very little) spacing between the discrete single-particle energy states.
  - One problem is our assumption of a perfect detector. Due to the continuum for massless particles, our process could, for example, create some extra very low energy ("soft") particles that will not be detected. We'll fix this now.
  - The other problem is that in the massless limit, the self-energy's derivative is ill defined; the pole at $k^2 = -m^2$ in the Lehmann-Källén form of the propagator merges with the branch point at $k^2 = -4m^2$ that we discovered in problem 15.1b, and is no longer a simple pole. We'll fix this in the next section.

# Splitting Final States

- Since there is now a continuum in energy, our final state particles can decay into two less energetic particles. The amplitude for this process, by the Feynman Rules, is:

$$\mathcal{T} = \frac{g}{k^2 + m^2}\mathcal{T}$$

- This term can diverge! To see the consequences of this, we calculate the cross-section.
  - We know how to do this, but our detector won't be able to tell the difference between a particle splitting or not splitting. So, we have to add the probabilities for the two events, which are distinguishable in principle.

# Splitting Final States

- After some math, we find:

$$|\mathcal{T}|^2_{\text{obs}} = |\mathcal{T}|^2 \left[ 1 + \frac{g^2}{(k^2+m^2)^2} (2\pi)^{d-1} \; 2\omega \; \delta^{d-1}(k_1 + k_2 - k) \frac{1}{2} \widetilde{dk}_1 \widetilde{dk}_2 + \ldots \right]$$

- Setting m --> 0, we have

$$\frac{\widetilde{dk_1}\widetilde{dk_2}}{(k^2)^2} \propto \frac{d\omega_1}{\omega_1^{5-d}} \frac{d\omega_2}{\omega_2^{5-d}} \frac{d\theta}{\theta^{7-d}}$$

- As we integrate over all omega and theta, this will clearly diverge at the low end (for low dimensionality)!
  - For dimensionality 6 or less, we have a divergence at low θ, corresponding to (nearly) collinear particles.
  - For dimensionality 4 or less, we have the additional problem of a divergence at low ω, corresponding to soft particles.

# (Nearly) Collinear Particles

- Let us assume that our detector can't tell the difference between two particles if the angle between their spatial momenta is smaller than $\delta$.

- Now we go back to our amplitude, do the math, and take the low mass limit.
  - The theta integral should range from 0 to $\delta$, since we can tell the difference between multiple particles at $\theta > \delta$.
  - Also multiply by four, because any one of the four particles could split.

# Conclusions

- The result is the exact scattering amplitude in six spacetime dimensions.
  - To be clear: there is an additional problem with soft particles. These particles are not necessarily collinear, but have too little energy to be detected. We already showed that this does not lead to a divergence in 6 spacetime dimensions, so no issue for $\varphi^3$ theory.

$$|\mathcal{T}|^2_{\text{obs}} = |\mathcal{T}_0|^2 \left[1 - \alpha\left(\frac{3}{2}\ln(s/m^2) + \frac{1}{3}\ln(1/\delta^2) + O(m^0)\right) + O(\alpha^2)\right]$$

- From this we conclude:
  - If $\delta$ is very small (ie our detector is very good), we will have to calculate high order corrections.
    - It seems like we're being punished for having a good detector, as a small $\delta$ decreases our amplitude. What's going on?
    - Consider the tree-level $\varphi^3$ vertex. With a good detector, we wouldn't count this toward the $\varphi\varphi \dashrightarrow \varphi\varphi$ scattering amplitude. With a crappy detector, there's a chance that the outgoing $\varphi$ actually masks two collinear outgoing particles, and so this diagram contributes a little bit, increasing the amplitude.
  - We still have this dratted term of $s/m^2$, which diverges in the massless limit. We'll deal with this in the next chapter.