QFT

Chapter 21: The Quantum Action

Overview

- Recall that we calculated the n-point vertex function by summing the skeleton diagrams
  - The skeleton diagrams had to be simple (no corrections), but we took each component of the simple diagram as being exact (all corrections included).

- We then drew all the tree-level processes for the vertex at hand, using all possible n-point vertices.

- But this is rather complicated! Couldn't we cook up an action, which would imply the tree-level diagrams?

Effective Action

- The action that directly yields the tree-level diagrams needed for the skeleton expansion is the following:

$$\begin{aligned}\Gamma(\phi) &= -\tfrac{1}{2}\int \tfrac{d^d k}{(2\pi)^d}\tilde{\phi}(-k)\left(k^2+m^2-\Pi(k^2)\right)\tilde{\phi}(k) \\ &\quad +\textstyle\sum_3^\infty \tfrac{1}{n!}\int \tfrac{d^d k_1}{(2\pi)^d}\ldots\tfrac{d^d k_2}{(2\pi)^d}(2\pi)^d\delta^d(k_1+\ldots+k_n) \\ &\quad \times \mathbf{V}_n(k_1,\ldots,k_n)\tilde{\phi}(k_1)\ldots\tilde{\phi}(k_n)\end{aligned}$$

- To reiterate: this is a bit complicated, but the advantage is that the resulting tree-level diagrams give the complete scattering amplitude of the original theory.

- Our next step is to figure out the relationship between the action (from ch. 9) and the effective action.

Old Action & New Action

- Recall how the action (S) is related to the sum of the connected diagrams (W):

$$Z(J) = \int \mathcal{D}\phi \exp\left[iS(\phi) + i\int d^d x J\phi\right] = exp[iW(J)]$$

- Can we do the same thing for the new (effective) action?

$$Z_\Gamma(J) = \int \mathcal{D}\phi \exp\left[i\left(\Gamma(\phi) + \int d^d x J\phi\right)\right] = exp[iW_\Gamma(J)]$$

- Well that sort of worked:
  - W_Γ is sum of connected diagrams in which each line represents the exact propagator and each n-point vertex represents the exact 1PI vertex.
  - But, we still have loop diagrams, and we no longer want them. In other words, W_Γ = W if we could get rid of all corrections to W_Γ.

Isolating Tree-Diagrams

- So, we need to keep only the tree diagrams. To do this, we'll use a cool trick – let's put the factors of ħ back.

$$Z_\Gamma(J) = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\left(\Gamma(\phi) + \int d^d x J\phi\right)\right] = exp[iW_\Gamma(J)]$$

- This gives propagators a factor of ħ, vertices and sources a factor of 1/ħ. The overall factor of ħ is $\hbar^{P-V-E}$
  - The reasoning as the same (but opposite) as the factors of i in chapter 9, see page 60.

- If we take $\hbar \rightarrow 0$, the dominant term will be that with P-V-E minimized. Now P-V-E = L-1, where L is the number of loops. So the dominant term will be that with the minimum number of loops, ie the tree diagram.
  - Why is P-V-E = L-1? Follow the degrees of freedom, and remember that only n-1 external lines are free (since [total] momentum is conserved).

Performing the Path Integral

- Hence, W_Γ = W in this limit. We now have:

$$Z_\Gamma(J) = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\left(\Gamma(\phi) + \int d^d x J\phi\right)\right] = exp[iW_\Gamma(J)]$$

where we take the limit as ħ→ 0.

- Now let's actually do the path integral
  - Method of stationary phase: find the point at which the exponent is stationary: this is given by the solution of the equation of motion:

$$\frac{\delta}{\delta\phi(x)}\Gamma(\phi) = -J(x)$$

  - Let's call the solution

$$\phi_J(x)$$

Conclusions

- Combining our results (prbm. 21.1), we have that

$$W(J) = \Gamma(\phi_J) + \int d^d x J \phi_J$$

which shows the relationships between the action and the effective action.

- A little calculus shows that, further,

$$\langle 0|\phi(x)|0\rangle_J = \phi_J(x)$$