QFT

Unit 2: Lorentz Invariance

Why are we talking about this?

- This is a bit of a departure from chapter 1!
- But, we want to impose relativity on our QFT from chapter 1. Lorentz Invariance allows us to formalize relativity.
- You've probably seen Lorentz Invariance before, but still go through this section very carefully.
  - New notation
  - The mathematical formalism for our "Lorentz Group" will be very important, and also a representative example of the group theory that we must deal with in QFT.

The Metric

- The metric, and therefore the interval, must be invariant (as observed before). Mathematically, this means:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

- We'll take the metric to be:

$$g_{ab} = \mathrm{diag}(-1, 1, 1, 1)$$

- I assume you're already familiar with the idea of the metric and with index notation; I will not go through those here.

The Lorentz Group

- The Lorentz Transformations form a group. Why?
  - Closure
  - Identity
  - Invertibility
  - Associativity

  We'll prove these statements separately.

- This group has the mathematical structure O(1,3).
  - O for orthogonal, i.e. the transformation will be linear. (if it weren't, couldn't write four-dimensional Lorentz transformation as one 4x4 matrix).
  - (1, 3) for time, space dimensions.

Infinitesimal Lorentz Transformations

- The infinitesimal Lorentz Transformation is given by:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$$

where this last term turns out to be antisymmetric (see problem 2.1)

This last term could be:

- A rotation of angle θ, where $\delta\omega_{ij} = -\varepsilon_{ijk}\hat{n}_k\delta\theta$
- A boost of rapidity η, where $\delta\omega_{ij} = \hat{n}_i\delta\eta$

Infinitesimal Lorentz Transformations

- Some Lorentz Transformations are formed by doing "many" infinitesimal ones.
- These will have the property of being proper and orthochronous
  - Proper: determinant = 1;
    - Improper: determinant = -1.
    - Determinants must be 1 or -1 (this follows from Srednicki 2.5, which I derive in problem 2.10).
  - Orthochronous: $\Lambda^0{}_0 \geq 1$
    - Non-orthochronous: $\Lambda^0{}_0 \leq -1$
    - $\Lambda^0{}_0$ must be one of these
      (follows from def'n of group, just plug in $\mu = \nu = \rho = \sigma = 0$)

Propriety & Orthochronaity

- Propriety and orthochronaity form subgroups of their own (proven separately).
  - Propriety has the structure SO(1, 3), where S is for special (determinant 1)
  - Orthochronaity has the structure O*(1, 3).
  - The union of the subgroups, which is obviously a subgroup itself, has the structure SO*(1, 3)

Why is this important?

- If QFT is to be consistent with relativity, it should be invariant under proper, orthochronous Lorentz Transformations.

- That means we should be able to replace x with Λx at any time without changing anything.

What about non-proper or non-orthochronous Lorentz Transformations?

- In QFT, we won't care about these in general.
  - They correspond to discrete transformations

- But, there are two interesting cases.
  - Parity, P = diag(1, -1, -1, -1)
  - Time Reversal, T = diag(-1, 1, 1, 1)
  - These should be symmetries also, though we treat them separately

Why do we care about some Lorentz Symmetries and not others?

- Noether's Theorem: every symmetry has a conservation law.
  - Infinitesimal (proper & orthochronous) Lorentz Transformations lead to:
    - Rotations, which conserve **angular momentum**
    - Boosts, which conserve the quantity $\mathbf{t(p_x) - r_xE}$
  - Time-reversal leads to **conservation of energy**. This happens to be a Lorentz Transformation also.
  - Parity leads to **conservation of momentum**. This happens to be a Lorentz Transformation also

- Invariance under other Lorentz Transformations does not have to be enforced, because these transformations do not lead to valid conservation laws.

Summary

- The Lorentz group is a mathematical object defined by the condition:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

- In QFT, the “Lorentz Group” is restricted to the proper, orthochronous subgroup, since these are the physical symmetries we expect.

- The time-reversal and parity operators happen to be Lorentz matrices also, but we treat them separately

Unitary Operators

- "In quantum theory, symmetries are represented by unitary (or anti-unitary) operators"
  - In this case, $U(\Lambda)$

- A Lorentz transformation is:

$$\phi_a(x) \to U_{ab}(\Lambda)\phi_b(\Lambda^{-1}x)$$

Peskin & Schroeder 3.8

- U is defined mathematically as:

$$U(\Lambda)^{-1}PU(\Lambda) = \Lambda P$$

- There's no new insight here. We can do this, so we do.

- These operators must obey composition:

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$$

Generators of a Group

- We' ve seen the idea of a generator before: Momentum is the Generator of Translation. Let' s review the proof of this:

$$f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \nabla^n f(x)$$

$$p = \frac{\hbar}{i}\nabla \implies \nabla^n = \left(\frac{ip}{\hbar}\right)^n$$

$$f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{ix_0 p}{\hbar}\right)^n f(x)$$

$$\boxed{f(x+x_0) = exp\left(\frac{ix_0 p}{\hbar}\right) f(x)}$$

Generator of the Lorentz Group

- The Lorentz Group has a generator too, though it's less intuitive. It is $M^{\mu\nu}$, where
  - M is defined to be antisymmetric
  - M is defined to be Hermitian
  - The "condition to be a generator" is satisfied:

$$U(1+\delta\omega) = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}$$

  - The commutation relations work out to be:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\,(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho})$$

(see problem 2.3)

Generators of the Lorentz Group

- We noted before that the Lorentz Group was made up of boosts and rotations
  - The angular momentum operator (generator of rotation) is $J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}$
  - The "boost operator" (generator of boosts) is
    $$K_i = M^{i0}$$
  - Srednicki then derives a bunch of commutation relations (see problems 2.4, 2.6, 2.7).

The Poincaré Group

- If we combine the Lorentz Group and the Translation Group, we get the Poincaré Group.
- Poincaré Group: the group of isometries of Minkowski Spacetime
  - Isometry means distance preserving, ie the interval is preserved
  - As noted before, we're interested in only the continuous Poincaré Group – ie all translations + proper and orthochronous Lorentz transformations
- The generators of the Translation and Lorentz Groups define the Lie Algebra of the Poincaré Group
  - Lie Algebra: generalized vector space (in this case, space of the generators) over a field ($R^{1,3}$ in this case) with a Lie Bracket (in physics, usually commutation) and certain other axioms, which are automatically met if your Lie Bracket is commutation.

Spacetime Translation Operator

- Time evolution is governed in quantum mechanics by the following:

$$e^{iHt/\hbar}\phi(x,0)e^{-iHt/\hbar} = \phi(x,t)$$

- Now we can "fix" the translation operator:

$$T(a) = exp\left(-iP^{\mu}a_{\mu}/\hbar\right)$$

Quantum Scalar Fields under a Lorentz Transformation

- What happens? Srednicki proves that we expect the following (2.26, rewritten slightly):

$$\phi(\Lambda x) = U(\Lambda)\phi(x)U^{-1}(\Lambda)$$

where derivatives carry vector indices that transform in the appropriate way.

This is the key result of the section: to impose a Lorentz Transformation, we don't have to change the arguments and dependency variables of everything. We just have to use these two operators as shown.