Unit 2: Lorentz Invariance
Why are we talking about this?

- This is a bit of a departure from chapter 1!

- But, we want to impose relativity on our QFT from chapter 1. Lorentz Invariance allows us to formalize relativity.

- You’ve probably seen Lorentz Invariance before, but still go through this section very carefully.
  - New notation
  - The mathematical formalism for our “Lorentz Group” will be very important, and also a representative example of the group theory that we must deal with in QFT.
The Metric

- The metric, and therefore the interval, must be invariant (as observed before). Mathematically, this means:

\[ g_{\mu \nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho \sigma} \]

- We’ll take the metric to be:

\[ g_{ab} = \text{diag}(-1, 1, 1, 1) \]

- I assume you’re already familiar with the idea of the metric and with index notation; I will not go through those here.
The Lorentz Group

- The Lorentz Transformations form a group. Why?
  - Closure
  - Identity
  - Invertibility
  - Associativity
    We’ll prove these statements separately.

- This group has the mathematical structure $O(1,3)$.
  - $O$ for orthogonal, i.e. the transformation will be linear.
    (if it weren’t, couldn’t write four-dimensional Lorentz transformation as one 4x4 matrix).
  - $(1, 3)$ for time, space dimensions.
Infinitesimal Lorentz Transformations

- The infinitesimal Lorentz Transformation is given by:

\[ \Lambda^\mu_\nu = \delta^\mu_\nu + \delta \omega^\mu_\nu \]

where this last term turns out to be antisymmetric (see problem 2.1)

This last term could be:
- A rotation of angle \( \theta \), where \( \delta \omega_{ij} = -\varepsilon_{ijk} \hat{n}_k \delta \theta \)
- A boost of rapidity \( \eta \), where \( \delta \omega_{ij} = \hat{n}_i \delta \eta \)
Infinitesimal Lorentz Transformations

Some Lorentz Transformations are formed by doing “many” infinitesimal ones.

These will have the property of being proper and orthochronous

- **Proper**: determinant = 1;
  - Improper: determinant = -1.
  - Determinants must be 1 or -1 (this follows from Srednicki 2.5, which I derive in problem 2.10).

- **Orthochronous**: $\Lambda^0_0 \geq 1$
  - Non-orthochronous: $\Lambda^0_0 \leq -1$
  - $\Lambda^0_0$ must be one of these
    (follows from def’ n of group, just plug in $\mu = \nu = \rho = \sigma = 0$)
Propriety & Orthochohraity

- Propriety and orthochohraity form subgroups of their own (proven separately).
  - Propriety has the structure SO(1, 3), where S is for special (determinant 1)
  - Orthochohraity has the structure O*(1, 3).
  - The union of the subgroups, which is obviously a subgroup itself, has the structure SO*(1, 3)
Why is this important?

- If QFT is to be consistent with relativity, it should be invariant under proper, orthochronous Lorentz Transformations.

- That means we should be able to replace $x$ with $\Lambda x$ at any time without changing anything.
What about non-proper or non-orthochronous Lorentz Transformations?

- In QFT, we won’t care about these in general.
  - They correspond to discrete transformations

- But, there are two interesting cases.
  - Parity, $P = \text{diag}(1, -1, -1, -1)$
  - Time Reversal, $T = \text{diag}(-1, 1, 1, 1)$
  - These should be symmetries also, though we treat them separately
Why do we care about some Lorentz Symmetries and not others?

- Noether’s Theorem: every symmetry has a conservation law.
  - Infinitesimal (proper & orthochronous) Lorentz Transformations lead to:
    - Rotations, which conserve angular momentum
    - Boosts, which conserve the quantity \( t(p_x) - r_x E \)
  - Time-reversal leads to conservation of energy. This happens to be a Lorentz Transformation also.
  - Parity leads to conservation of momentum. This happens to be a Lorentz Transformation also

- Invariance under other Lorentz Transformations does not have to be enforced, because these transformations do not lead to valid conservation laws.
Summary

- The Lorentz group is a mathematical object defined by the condition:
  \[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \]

- In QFT, the “Lorentz Group” is restricted to the proper, orthochronous subgroup, since these are the physical symmetries we expect.

- The time-reversal and parity operators happen to be Lorentz matrices also, but we treat them separately.
Unitary Operators

- “In quantum theory, symmetries are represented by unitary (or anti-unitary) operators”
  - In this case, $U(\Lambda)$

- A Lorentz transformation is:
  \[ \phi_a(x) \rightarrow U_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \]

- $U$ is defined mathematically as:
  \[ U(\Lambda)^{-1}PUU(\Lambda) = \Lambda P \]

- There’s no new insight here. We can do this, so we do.

- These operators must obey composition:
  \[ U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda) \]
Generators of a Group

We’ve seen the idea of a generator before: Momentum is the Generator of Translation. Let’s review the proof of this:

\[ f(x + x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \nabla^n f(x) \]

\[ p = \frac{\hbar}{i} \nabla \quad \implies \quad \nabla^n = \left( \frac{ip}{\hbar} \right)^n \]

\[ f(x + x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{ix_0 p}{\hbar} \right)^n f(x) \]

\[ f(x + x_0) = \exp \left( \frac{ix_0 p}{\hbar} \right) f(x) \]
Generator of the Lorentz Group

The Lorentz Group has a generator too, though it’s less intuitive. It is $M^{\mu \nu}$, where

- $M$ is defined to be antisymmetric
- $M$ is defined to be Hermitian
- The “condition to be a generator” is satisfied:

$$U(1 + \delta \omega) = I + \frac{i}{2\hbar} \delta \omega_{\mu \nu} M^{\mu \nu}$$

The commutation relations work out to be:

$$[M^{\mu \nu}, M^{\rho \sigma}] = i\hbar \left( g^{\mu \rho} M^{\nu \sigma} - g^{\nu \rho} M^{\mu \sigma} - g^{\mu \sigma} M^{\nu \rho} + g^{\nu \sigma} M^{\mu \rho} \right)$$

(see problem 2.3)
Generators of the Lorentz Group

- We noted before that the Lorentz Group was made up of boosts and rotations
  - The angular momentum operator (generator of rotation) is
    \[ J_i = \frac{1}{2} \varepsilon_{ijk} M^{jk} \]
  - The “boost operator” (generator of boosts) is
    \[ K_i = M^{i0} \]
  - Srednicki then derives a bunch of commutation relations (see problems 2.4, 2.6, 2.7).
The Poincaré Group

- If we combine the Lorentz Group and the Translation Group, we get the Poincaré Group.

- Poincaré Group: the group of isometries of Minkowski Spacetime
  - Isometry means distance preserving, ie the interval is preserved
  - As noted before, we’re interested in only the continuous Poincaré Group – ie all translations + proper and orthochronous Lorentz transformations

- The generators of the Translation and Lorentz Groups define the Lie Algebra of the Poincaré Group
  - Lie Algebra: generalized vector space (in this case, space of the generators) over a field ($\mathbb{R}^{1,3}$ in this case) with a Lie Bracket (in physics, usually commutation) and certain other axioms, which are automatically met if your Lie Bracket is commutation.
Spacetime Translation Operator

- Time evolution is governed in quantum mechanics by the following:

\[ e^{iHt/\hbar} \phi(x, 0) e^{-iHt/\hbar} = \phi(x, t) \]

- Now we can “fix” the translation operator:

\[ T(a) = \exp \left( -iP^\mu a_\mu / \hbar \right) \]
Quantum Scalar Fields under a Lorentz Transformation

What happens? Srednicki proves that we expect the following (2.26, rewritten slightly):

\[ \phi(\Lambda x) = U(\Lambda)\phi(x)U^{-1}(\Lambda) \]

where derivatives carry vector indices that transform in the appropriate way.

This is the key result of the section: to impose a Lorentz Transformation, we don’t have to change the arguments and dependency variables of everything. We just have to use these two operators as shown.