QFT

Chapter 15: The One-Loop Correction in Lehmann-Källén Form

Overview

- It is not obvious that the Lehmann-Källér Propagator for φ[3] theory, from chapter 14:

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} ds\ \rho(s) \frac{1}{k^2 + s - i\epsilon}$$

  is equivalent to the exact propagator from chapter 15:

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)}$$

$$\Pi(k^2) = \frac{\alpha}{2} \int_0^1 D\ ln(D/D_0) - \frac{\alpha}{12}(k^2 + m^2) + O(\alpha^2)$$

- In this short section, we search for relationships between Π and ρ that result from requiring these two forms to be equal.

L-K vs. Perturbative, 1

- The imaginary part of the Lehmann-Källér Propagator is given by:

$$Im\ \tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2+m^2) + \pi\int_{4m^2}^{\infty} ds\rho(s)\delta(k^2+s)$$

$$Im\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2+m^2) + \pi\rho(-k^2)$$

- Now take s = $-k^2 \geq 4m^2$. Then:

$$Im\ \tilde{\mathbf{\Delta}}(-s) = \pi\rho(s)$$

L-K vs. Perturbative, 2

- Now we'll assume that the imaginary part of the self-energy is zero. Using the result from chapter 15, this means that

$$Im\ \tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2 - i\Pi(k^2))$$

- Imposing the boundary conditions, $\Pi = 0$

$$Im\ \tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2)$$

- Now remember this result from the previous page:

$$Im\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2) + \pi\rho(-k^2)$$

- We impose reconciliation between the two forms by setting $\rho(-k^2) = 0$ when $\mathrm{Im}\ \Pi(k^2) = 0$.

L-K vs. Perturbative, 3

- What if Im $\Pi(k^2) \neq 0$? Then the perturbative form becomes:

$$Im\ \tilde{\mathbf{\Delta}}(k^2) = \frac{Im\ (k^2)}{(k^2 + m^2 - Re\ \Pi(k^2))^2 + (Im\ \Pi(k^2))^2}$$

- Comparing this to the result from the Lehmann-Källér form:

$$Im\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2) + \pi\rho(-k^2)$$

We find that:

$$\pi\rho(s) = \frac{Im\ (-s)}{(-s + m^2 - Re\ \Pi(-s))^2 + (Im\ \Pi(-s))^2}$$

- Further, since $\rho(s) = 0$ if $s < 4m^2$, we have:

  Im $\Pi(-s) = 0$ for $s < 4m^2$

L-K vs. Perturbative, 4

- So these two forms are equivalent, provided we have:

$$\pi\rho(s) = \frac{Im\ (-s)}{(-s+m^2 - Re\ \Pi(-s))^2 + (Im\ \Pi(-s))^2}$$

$\rho(-k^2) = 0$ when $\mathrm{Im}\ \Pi(k^2) = 0$

- This first form implies the following:

$$Im\ \Pi(k^2) = 0 \text{ for } k^2 > -4m^2$$

- To verify, let's go back to our results for φ^3 theory in chapter 14. The imaginary part is indeed zero at $k^2 > -4m^2$, so everything appears to be in order.

One Comment

- Srednicki presents this section as a “reconciliation” between the two forms of the exact propagator.
- However, Srednicki never proves that the real parts are equal if $\mathrm{Im}\,\Pi \neq 0$, so the proof is incomplete.
- Moreover, the proof is perhaps unnecessary, since he presented valid derivations for each separately. The point of this chapter is to find the relationship between spectral density and self-energy.

Principle Parts & Delta Functions

- Srednicki also uses a sort of shorthand that is perhaps worth presenting:

$$\frac{x}{x^2 + \epsilon^2} = P\frac{1}{x}$$

where P is called the principle part.

- He also uses this relationship, which is useful:

$$\frac{\epsilon}{x^2 + \epsilon^2} = \pi\delta(x)$$

where the delta function makes sense conceptually, and the normalization is found by integrating both sides over all space.