QFT

Chapter 14: Loop Corrections to the Propagator

Overview

- Here we turn to our next major topic: loop order corrections. We'll consider the effect on the propagator first.

- This has at least two advantages:
  - More accurate decay rates, cross-sections, etc.
  - But also surprisingly useful in its own right, for example, one argument for SUSY is that the Higgs propagator is unstable, but can be stabilized by loop-order supersymmetric corrections.

Exact Propagator, again

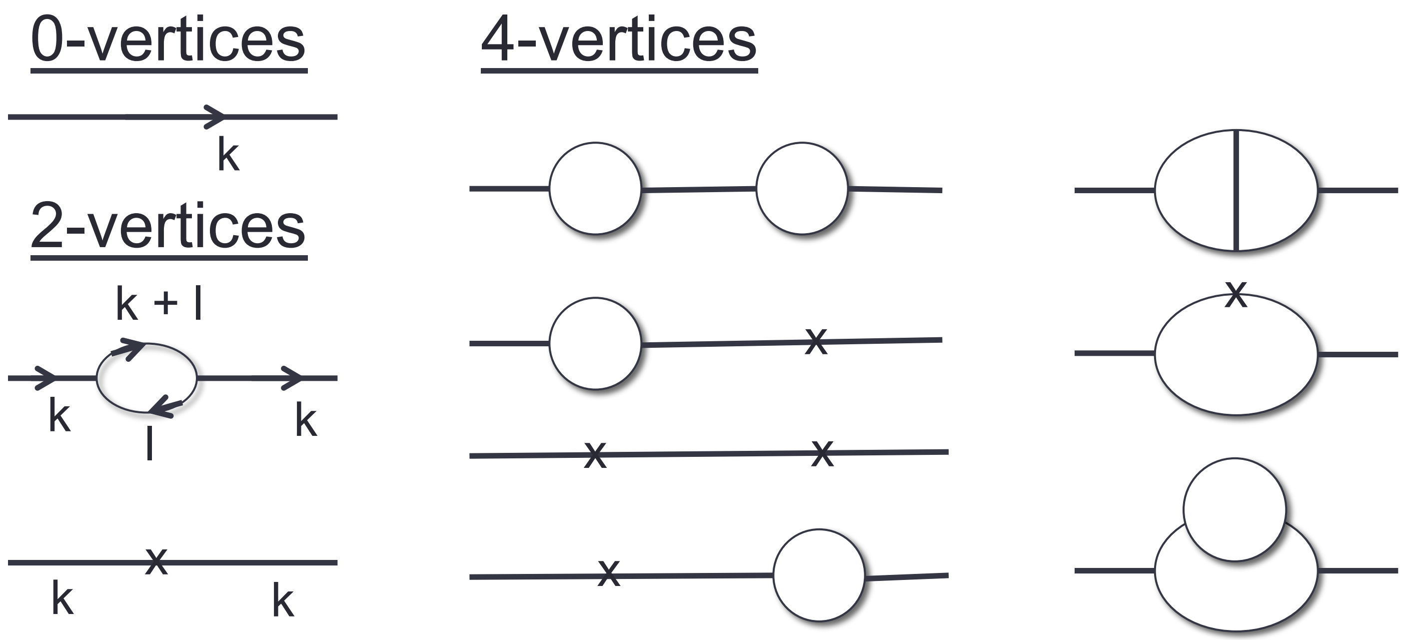
- Recall that the exact propagator is given by

$$\frac{1}{i}\Delta(x-y) = \delta_x \delta_y iW(J)\Big|_{J=0}$$

  - W is the sum over all connected diagrams.
  - If there are more than two sources, then imposing J = 0 will cause the term to vanish
  - If there are less than two sources, then the derivative will kill the term
  - So, we need to draw all diagrams with 2 sources (and then remove the sources). Summing all these diagrams will yield the exact propagator.

Exact Propagator, again

- We give the propagator to $O(g^4)$ in φ^3 theory:



Notes about these diagrams

- Note that the $O(g^2)$ diagrams also contribute to $O(g^4)$, since both vertex factors have higher-order corrections.
  - Srednicki is a little misleading on this point, since he includes only one such diagram in figure 14.3.

- Note also that the counterterm diagram (the one that looks like an x) is order g^2 – it therefore counts for two vertices.
  - This is because Srednicki tells us that $Z_k = 1 + O(g^2)$, and A, B are $Z_k - 1$.

Exact Propagator, perturbatively

- Let's now use the Feynman Rules to determine the exact propagator in momentum space. This will be perturbative, since we're summing over the diagrams.

- Note that:
  - The symmetry factor for the loop diagram is 2.
  - The vertex factor for the 3-point vertex is ig, since $Z_g = 1 + O(g^2)$.
  - These diagrams are for a propagator, not a scattering process, so the external lines get the free-field propagator factor, not the factor of 1 associated with external lines.
  - $\mathbf{\Delta}(k^2)$ refers to the exact propagator, while $\Delta(k^2)$ refers to the free-field propagator. This is rather confusing notation, but is consistent with the book.

Exact Propagator, perturbatively

- The result is that:

$$\frac{1}{i}\widetilde{\mathbf{\Delta}}(k^2) = \frac{1}{i}\widetilde{\Delta}(k^2) + \frac{1}{i}\widetilde{\Delta}(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\widetilde{\Delta}(k^2) + O(g^4)$$

where Π(x) represents the self-energy:

$$i\Pi(k^2) = \frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\tilde{\Delta}((l+k)^2)\tilde{\Delta}(l^2) - i(Ak^2 + Bm^2) + O(g^4)$$

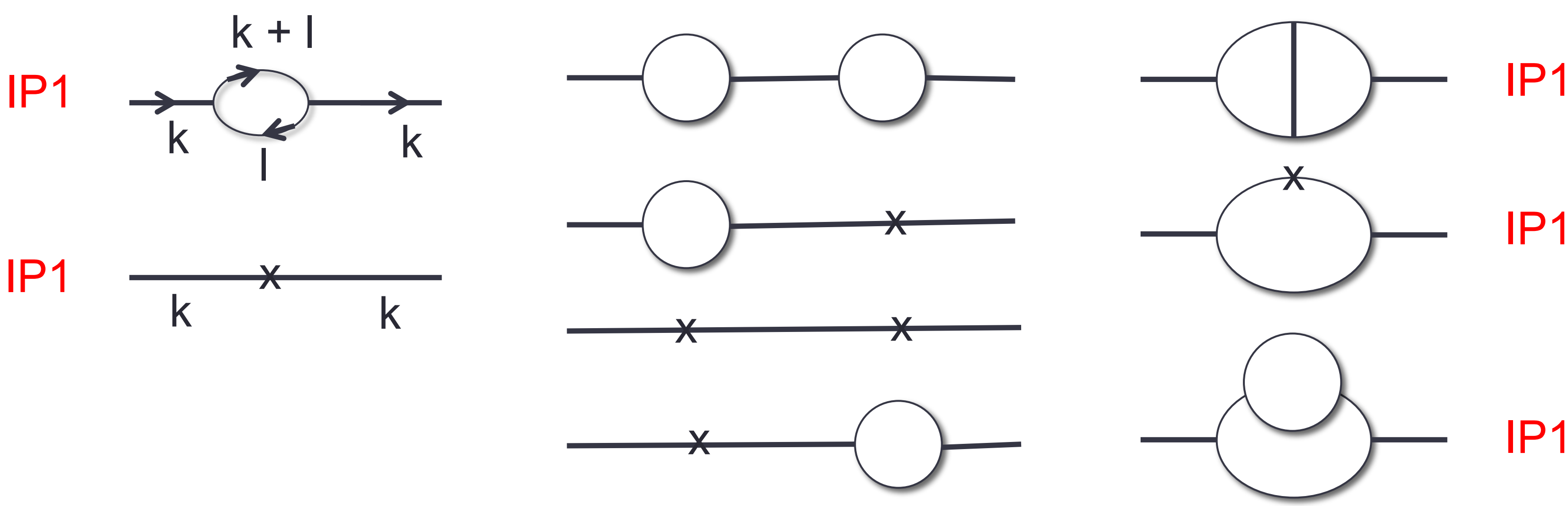
- Back to the Feynman Diagrams, let's define a *one-particle irreducible* (1PI) diagram to be one that is still connected after any one internal line is cut
  - Internal means not including the two external propagators. Srednicki is a little unclear on this.

1PI Diagrams

- Let's label which of these diagrams are 1PI diagrams, and which ones are not.

<u>2-vertices</u>

<u>4-vertices</u>



Exact Propagator, perturbatively

- Now, how can we expand to fourth order?
  - What we've done in the past is to go by vertices. We draw all the diagrams with four or fewer vertices, and keep the fourth-order contributions from each diagram.
    - The problem is that there are many diagrams!
  - A better option is to go by 1PI vertices. We'll define Π to be the sum of 1PI diagrams, and define the exact propagator to be given by:

$$\frac{1}{i}\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{i}\tilde{\Delta}(k^2)\left[1 + \Pi(k^2)\tilde{\Delta}(k^2) + \Big(\Pi(k^2)\tilde{\Delta}(k^2)\Big)^2 + \dots\right]$$

    - It is easy to see that this is identical to the vertex-expansion, though the terms are organized differently. All the 1PI diagrams (even the complicated ones with many vertices) are in the second term, while those diagrams with two 1PI components come later (even the simple ones with only 4 vertices)

Exact Propagator, perturbatively

- Another advantage is that this geometric series can be summed.

$$\begin{aligned} S &= 1 + x + x^2 + \ldots \\ xS &= x + x^2 + \ldots \\ \implies & \quad S - xS = 1 \\ \implies & \quad S = \tfrac{1}{1-x} \end{aligned}$$

- Applying this to our expansion, and using the explicit form of the free-field theory propagator that we found in chapter 8, this becomes:

$$\widetilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)}$$

- This is sort of a nice formula, similar to our result from chapter 10, but inclusive of loop-order corrections.
  - Also there are only 1PI diagrams that need to be summed over – we don't have to sum over every diagram with a vertex.

Exact Propagator, perturbatively

$$\widetilde{\mathbf{\Delta}}(\mathbf{k^2}) = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)}$$

- The Lehmann-Källén formula tells us that we should have a pole at $k^2 = -m^2$ with residue 1. This implies that:

  $\Pi(-m^2) = 0$

  $\Pi'(-m^2) = 0$, where the prime denotes the k^2 derivative.

- We will use these to fix the values of A and B.

$O(g^2)$ Corrections to Propagator

- We've now established that if we know Π, we can get the propagator. We also know that Π is given by:

$$i\Pi(k^2) = \frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\tilde{\Delta}((l+k)^2)\tilde{\Delta}(l^2) - i(Ak^2+Bm^2) + O(g^4)$$

and of course:

$$\widetilde{\Delta}(k^2) = \frac{1}{k^2+m^2-i\epsilon}$$

- But if d > 3, then this integral will diverge at large l
    - Imagine polar coordinates. In four dimensions (or higher), the integration measure will have a factor of l^3 (or higher). Doing the integral will give something of leading order log l (or higher), which diverges at large l.

$O(g^2)$ Corrections to Propagator

- For now, we'll restrict ourselves to $d < 4$ to avoid this problem. We saw in chapter 12 that we eventually want to work in $d = 6$ (for this φ^3 theory), so we can't avoid this problem forever.

- Even given this assumption, this integral is still very difficult to evaluate – and will get even harder for more complicated theories at higher orders.
  - We will therefore spend some time developing a "bag of tricks" which we can use to get through these types of integrals.

Trick #1: Feynman's Formula

- Feynman's Formula allows us to combine denominators:

$$\frac{1}{A_1 \ldots A_n} = \int dF_n (x_1 A_1 + \ldots + x_n A_n)^{-n}$$

- The integration measure is defined as:

$$\int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta(x_1 + \ldots + x_n - 1)$$

- The normalization is given by:

$$\int dF_n 1 = 1$$

Evaluating the Self-Energy, 1

- Recall that our definition of $\Pi(k^2)$ contains two (internal) propagators. We use Feynman's formula to simplify:

$$\tilde{\Delta}((k+\ell)^2)\tilde{\Delta}(\ell^2) = \frac{1}{(\ell^2+m^2)((\ell+k)^2+m^2)}$$

$$\tilde{\Delta}((k+\ell)^2)\tilde{\Delta}(\ell^2) = \int_0^1 dx\left[x((\ell+k)^2+m^2)+(1-x)(\ell^2+m^2)\right]^{-2}$$

- After some algebra, we obtain:

$$\tilde{\Delta}((k+\ell)^2)\tilde{\Delta}(\ell^2) = \int_0^1 dx\left[q^2+D\right]^{-2}$$

$$q = \ell + xk$$

$$D = x(1-x)k^2 + m^2$$

where we suppressed the epsilons for notational convenience

Evaluating the Self-Energy, 2

- Finally, let's change the integration variable for the self-energy from l to q.

- Writing the entire self-energy, we find:

$$i\Pi(k^2) = \frac{1}{2}(ig)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \left[q^2 + D\right]^{-2} - i(Ak^2 + Bm^2) + O(g^4)$$

where we have:

$$D = x(1-x)k^2 + m^2$$

Trick #2: Wick Rotation

- Let's think of the q_0 integral as being a contour integral in the complex plane. The Wick Rotation rotates the contour counterclockwise by 90°. This is allowed:
  - So long as we don't pass over any poles (we obviously won't in the case at hand, see figure 14.4)
  - So long as the integrand vanishes fast enough as the magnitude of q_0 diverges (this is the usual condition for contour integrals)

- This is imposed by the following relation:

$$q^0 = i\bar{q}_d \qquad q_j = \bar{q}_j$$

- The advantage of this is that we can set $\varepsilon = 0$, since the singularity is no longer along the axis of integration.

Evaluating the Self-Energy, 3

- The self-energy is given by:

$$\Pi(k^2) = \frac{1}{2}g^2 \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} - Ak^2 - Bm^2 + O(g^4)$$

- Let's define part of this to be I(k^2):

$$I(k^2) = \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2}$$

Trick #3: Second Derivative

- Before moving on, let's take the second derivative of this. This has two advantages:
  - No need to explicitly compute A and B, we can just proceed and then take two integrals when we're done. This will save us a few steps.
  - The integral will now be finite for $d < 8$, rather than $d < 4$. This is a huge help, since the case of greatest interest ($d = 6$) is now included.
    - How did that happen? What's really going on is that we're Taylor expanding $I(k^2)$, and then choosing A and B to cancel the first two terms. The remaining terms must be finite, which is the case for $d < 8$.
    - Of course, this means that Z_φ and Z_m are formally infinite (this is similar to Y, as we saw in chapter 9). But that's OK, since these are not directly measurable parameters; all the formally infinite numbers will cancel (by design) when computing the self-energy.

- The point of this trick is that we know we can analytically continue our results to the region $4 \leq d \leq 8$.
  - We could certainly get the result by using this trick, but we may prefer to obtain the result through a different method.

Renormalizability

- What if this hadn't worked out? Or what if we lived in a 9-dimensional universe? How then would we deal with an infinite self-energy?

- This means that something is wrong with the theory – we call such theories nonrenormalizable. We'll discuss this more later.

- It turns out that φ^3 theory is renormalizable for $d < 7$. The problem with $d = 7$ is in the higher order corrections.
  - We've already shown, of course, that it's renormalizable for $d < 9$ up to $O(g^2)$

How to proceed?

- We currently have:

$$\Pi(k^2) = \frac{1}{2}g^2 \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} - Ak^2 - Bm^2 + O(g^4)$$

- Options:
  - Use trick #3 – take two derivatives, do the integral, then take two integrals and fit the boundary conditions.
  - Use Pauli-Villars Regularization, which is what we did in chapter 9. Multiply the free propagator by a factor of $\Lambda^2/(p^2 + \Lambda^2 - i\varepsilon)$, fit the boundary conditions, and allow $\Lambda \to \infty$.
  - Use dimensional regularization: just do the integral directly, and analytically continue the result to d = 6, which we know is OK due to trick #3.

- We choose option #3.

Trick #4: Shift Dimensionality to mu tilde

- Now let's do the integral. The result is:

$$I(k^2) = \frac{\Gamma(2-d/2)}{(4\pi)^3} \int_0^1 dx D \left(\frac{4\pi}{D}\right)^{3-d/2}$$

- Recall that the mass dimension of g depends on the dimensionality of the problem. This is quickly going to get confusing, so let's shift $g \to g\tilde{\mu}^{3-d/2}$. Now g is always dimensionless, and mu is always not.
  - This is not a general result: the definition of mu depends on the theory being considered.
  - Further, mu is not a "real" parameter, so nothing measurable should depend on it.

Evaluating the Self-Energy, 4

- Defining α = g²/(4π)³ and ε = 6-d we have:

$$\Pi(k^2) = \frac{1}{2}\alpha\Gamma\left(-1+\frac{\varepsilon}{2}\right)\int_0^1 dx D\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon/2} - Ak^2 - Bm^2 + O(\alpha^2)$$

- At this point there are no more tricks, all that remains is to do the calculus. The result is given by:

$$\Pi(k^2) = \frac{1}{2}\alpha\left[(3-\pi\sqrt{3})k^2 + (3-2\pi\sqrt{3})m^2 + 2k^2(1+4m^2/k^2)^{3/2}tanh^{-1}\left(\frac{1}{\sqrt{1+4m^2/k^2}}\right)\right] + O(\alpha^2)$$

- We already found the exact propagator, but we can also write it like this:

$$\tilde{\boldsymbol{\Delta}}(k^2) = \left(\frac{1}{1-\Pi(k^2)/(k^2+m^2)}\right)\frac{1}{k^2+m^2-i\epsilon}$$

Conclusions

- The only unsettling thing about our result is the way that the real part of the self energy increases logarithmically with k^2 as k^2 is large. We'll address the meaning of this later.

- Nonetheless, we have a formula for the exact propagator in terms of the self-energy, and we calculated the self-energy to second order.