QFT

Unit 13: The Lehmann-Källén form of the Exact Propagator

Overview

- In this short section, we pause briefly to derive an exact form for the momentum-space propagator, which may be useful later on.
- Next time, we turn to our next major topic: loop corrections as they apply to the field theory we’ve developed previously.

Momentum-Space Propagator in Free Field Theory

- We defined this in eqn. 8.11 for position-space. Now let's take the Fourier transform to get this in momentum-space:

$$\widetilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon}$$

- Now our job is to get something analogous for interacting field theory:

Derivation of Lehmann-Källén, 1

- We start with the definition of the propagator:

$$\Delta(x-y) = i\langle 0|T\phi(x)\phi(y)|0\rangle$$

- For simplicity, we'll assume these are already time-ordered, and insert a complete set of states:

$$\Delta(x-y) = i\sum_j \langle 0|\phi(x)|j\rangle\langle j|\phi(y)|0\rangle$$

- Now consider that there are three types of states:
  - The ground state
  - One-particle states
  - Multi-particle states.

Derivation of Lehmann-Källén, 2

- Accounting for these three types of states, we have:

$$\Delta(x-y) = i\left[\langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle + \int \widetilde{dk}\langle 0|\phi(x)|k\rangle\langle k|\phi(y)|0\rangle + \sum_n \int \widetilde{dk}\langle 0|\phi(x)|k,n\rangle\langle k,n|\phi(y)|0\rangle\right]$$

  - In the last state, *n* refers to all the junk needed to specify the state, and the sum indicates to integrate or sum over all of it.

- The first term can be killed by our renormalization scheme. Remember that we set this to zero because we didn't want the creation operator "inside" φ to create some linear combination of the ground state.

Derivation of Lehmann-Källén, 3

- Now we have:

$$\Delta(x-y) = i\left[\int \widetilde{dk}\langle 0|\phi(x)|k\rangle\langle k|\phi(y)|0\rangle + \sum_n \int \widetilde{dk}\langle 0|\phi(x)|k,n\rangle\langle k,n|\phi(y)|0\rangle\right]$$

- Again, we can use the renormalization scheme. Separating out the time dependence, we get

$$e^{-ipx}\langle p|\phi(0)|0\rangle$$

- This is a Lorentz-invariant function of p, which must be $p^2 = -m^2$, which is a constant – so the entire thing must be a constant. We want the constant to equal one, just as in free-field theory, where it yields a correctly normalized state.

Derivation of Lehmann-Källén, 4

- Now we have:

$$\Delta(x-y) = i\left[\int \widetilde{dk} e^{ik(x-y)} + \sum_n \int \widetilde{dk} \langle 0|\phi(x)|k,n\rangle\langle k,n|\phi(y)|0\rangle\right]$$

- Separating out the time-dependence gives:

$$\Delta(x-y) = i\left[\int \widetilde{dk} e^{ik(x-y)} + \sum_n \int \widetilde{dk} e^{ik(x-y)} |\langle k,n|\phi(y)|0\rangle|^2\right]$$

Spectral Density

- Typically in physics, we use spectral density to show the frequency-dependence of Energy

- Here, we make the definition:

$$\rho(s) = \sum_n |\langle k, n|\phi(0)|0\rangle|^2 \delta(s - M^2)$$

- It may not be immediately obvious that this is spectral density, but early signs look good.
  - This is the probability of transition from a one particle state to a multi particle state, given that the multi-particle state has a total mass M, which is at least 2m (and probably more due to relative momentum).
  - M is part of the "junk" specified by n.
  - Using s rather than M^2 is just a convention. In particular, it has nothing to do with the Mandelstam variable s.

Derivation of Lehmann-Källén, 5

- Now we have:

$$\Delta(x-y) = i\left[\widetilde{dk}e^{ik(x-y)} + \int_{4m^2}^{\infty} ds\ \rho(s)\int \widetilde{dk}e^{ik(x-y)}\right]$$

- But this is only for $x^0 > y^0$. If we do this for $y^0 > x^0$, then get the same thing reversed. We combine these results with theta functions.
  - But this seems familiar! We did this in chapter 8. Referring back to that section, we find a more compact way to write this…

Derivation of Lehmann-Källén, 6

$$\Delta(x-y) = \int \frac{d^dk}{(2\pi)^d} e^{ik(x-y)} \left[\frac{1}{k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\ \rho(s) \frac{1}{k^2+s-i\epsilon} \right]$$

- Taking the d-dimensional Fourier Transform:

$$\widetilde{\Delta}(k^2) = \frac{1}{k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\ \rho(s) \frac{1}{k^2+s-i\epsilon}$$

which is our result.