QFT

Unit 11: Cross Sections and Decay Rates

Decays and Collisions

- When it comes to elementary particles, there are only two things that ever really happen:
  - One particle decays into stuff
  - Two particles collide into each other.

- So, our goal is really to calculate two things:
  - Decay rates
  - Cross-sections (“collision rates”)

We’ll start with collisions, specializing to the case of two incoming and two outgoing particles.

Mandelstam Variables

- We're deeply concerned about the kinematics of the collision, so let's define three variables that will be useful:
  - $s = -(k_1 + k_2)^2 = -(k_1' - k_2')^2$
    - Center of mass energy squared
  - $t = -(k_1 - k_1')^2 = -(k_2 - k_2')^2$
    - Related to angle between k_1 and k_1'.
  - $u = -(k_1 - k_2')^2 = -(k_2 - k_1')^2$
- These follow the linear relation:

  $s + t + u = m_1^2 + m_2^2 + m_{1'}^2 + m_{2'}^2$
- Better yet, these are each **Lorentz-Invariant.**

Kinematics of the Collision

- Recall that all these particles are *on shell*, meaning that $k_i^2 = -m_i^2$.

- In the center-of-mass frame:
  - The initial vector three-momenta must sum to zero.
  - k_1 is defined to be along the +z-axis
  - The only other thing needed to define the initial state is the magnitude of k_1, which can be calculated from s.

- There are a few useful formulas that relate the Mandelstam variables to the kinematic details in various frames; Srednicki works these out. It's just algebra, so I won't go through it here.

Differential Scattering Cross Section

- Assumptions:
  - Whole experiment takes place in box of volume V
  - Whole experiment lasts for a long time T.
  - The number of outgoing particles will be arbitrary (to get the most general answer)

- We want the probability of a transition from i to f. This is given by: $P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$

  where $\langle f|i\rangle = (2\pi)^4\delta^4(k_{in} - k_{out})i\tau$

  - As for squaring the delta function – the second delta function is just $\delta^4(0)$, and $(2\pi)^4\delta^4(0) = VT$ (see eqn. 11.14)

Differential Scattering Cross Section, cntd.

- Remember also equation 9.3:

$$\langle k' | k \rangle = (2\pi)^3 2k^0 \delta^3(\vec{k}' - \vec{k})$$

  - The delta functions make sense; the first three terms are required since the integral will be with respect to dLIPS.

- Putting all this together, the probability per unit time is given by:

$$\dot{P} = \frac{(2\pi)^4 \delta^4(k_{in} - k_{out}) V |\tau|^2}{4E_1 E_2 V^2 \prod_{j=1}^{n'} 2k_j'^0 V}$$

Differential Scattering Cross Section, cntd.

- This result is the probability of decaying into a particle with momentum *exactly* k_1, k_2, etc.
  - Remember that we're in a box, so these are quantized (though infinite) and the probability for the lower states are nonzero.

- To remove the box requirement, we take:

$$\sum_{\vec{n}'_j} \rightarrow \frac{V}{(2\pi)^3} \int d^3 k'_j$$

- The result is:

$$\dot{P} = \frac{(2\pi)^4 \delta^4 (k_{in} - k_{out})}{4 E_1 E_2 V} |\tau|^2 \prod_{j=1}^{n'} \widetilde{dk}'_j$$

Differential Scattering Cross-Section, cntd.

- Remember what a cross-section is:
  - Effective area seen by particles that could potentially collide.
  - For many incoming particles, the cross section is proportional to the probability of interaction
    - Probability that a particle will interact = σ/V = dσ/dΩ (in limit of many particles)
    - Since we could potentially have many incoming particles, we have to divide by incident flux.

- Incident Flux:
  - Number of particles per unit volume, times their speed.
  - So, incident flux is $|k_1|/E_1 V$.

- The result is that the differential scattering cross section is:

$$d\sigma = \frac{1}{4|\vec{k}_1|_{CM}\sqrt{s}} |\tau|^2 d\mathrm{LIPS}_{n'}(k_1 + k_2)$$

$$\text{where } d\mathrm{LIPS}_{n'}(k) = (2\pi)^4 \delta^4 \left(k - \sum_{i=0}^{n'} k_i' \right) \prod_{j=1}^{n'} \widetilde{dk}_j'$$

Differential Cross-Sections for 2 to 2 Scattering

- If we specialize to the case with two outgoing particles, we can simplify. This is just calculus (though there are some tricks that are worth observing), so I quote the result:

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} \frac{|\vec{k}_1'|}{|\vec{k}_1|} |\tau|^2$$

- The problem is that this is not Lorentz-invariant. The Lorentz-invariant equivalent is:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{k}_1|^2} |\tau|^2$$

  - Remember that $|k_1|$ is given by a complicated function of s.
  - This can be converted to dσ/dΩ in a given frame by taking the differential of dt in that frame. This leads to complicated results, however – see problem 11.2 for an example.

Total Cross Section

- The total cross section is by integrating over the outgoing momenta and dividing by a symmetry factor:

$$\sigma = \frac{1}{\prod_i n'_i!} \int d\sigma$$

  - The symmetry factor is needed because the final state is labeled by an unordered list of the momenta, not an ordered list of them.

Cross-Section for φ^3 Theory

- We worked out in chapter 10 the matrix element for φ^3 theory (something messy). Writing it with Mandelstam variables, it cleans up nicely:

$$\tau = g^2 \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right] + O(g^4)$$

- At least one of s, t, and u cannot be linearly independent: it turns out here that both are (easiest to see in the CM frame). So, this becomes complicated again. To make sense, we expand around:
  - Non-relativistic limit: nearly isotropic distribution
  - Relativistic limit: highly-peaked in forward and backward directions.

Cross-Section for φ^3 Theory, cntd.

- These expansions can be integrated to find the total cross section. The result is again complicated, but <u>may depend</u> on Mandelstam variables (since these are frame-independent).

Decay

- Recall that the transition amplitudes are based on the LSZ formula, and the LSZ formula requires that particles be an exact eigenstate of the exact Hamiltonian
  - See chapter 5, where we made a big deal about the multi-particle states and creation-operators working the same way as those for the single-particle states
  - But this is not the case for a particle that can decay!

- We'll address this issue later (ch. 25). For the moment, let's assume that the LSZ formula still works.
  - All the analysis from before still holds (except the initial state is redefined). Realizing that dΓ = $\dot{P}$, we have:

$$\Gamma = \frac{1}{S}\int \frac{1}{2E_1}|\tau|^2 d\mathrm{LIPS}_{n'}(k_1)$$

  - Also, s is now just m^2, since there is no second particle.