



# QFT

## Unit 1: Relativistic Quantum Mechanics

# What's QFT?

- Relativity deals with things that are fast
- Quantum mechanics deals with things that are small
- QFT deals with things that are both small and fast
  - What's small and fast? Elementary particles!  
We will deal with particles directly here.

# The Schrödinger Equation

- Why can't we just stick with quantum mechanics?
  - It's not relativistic.
- Why not?

An axiom of QM is the Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = H |\psi, t\rangle$$

# The Schrödinger Equation

Let's try a spinless particle with no forces acting on it. In QM, the Schrödinger Equation

becomes

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \frac{1}{2m} \mathbf{p}^2 |\psi, t\rangle$$

In our case, we'll use the relativistic formula

in:

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} |\psi, t\rangle$$

This seems fine, but not useful to have an operator under a square root!

# The Schrödinger Equation

To make it useful, we'll expand. Let's expand around  $c = \infty$ , since anything divided by many powers of  $c$  will be negligible. Then,

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \left( mc^2 + \frac{1}{2m} \mathbf{p}^2 - \frac{1}{8c^2 m^3} \mathbf{p}^4 + \dots \right) |\psi, t\rangle$$

This is not consistent with relativity.

- Space and time treated differently
  - Time derivative only on left, once
  - Spatial derivative only on right, many times
- Nonlocal in space
  - Infinite number of spatial derivatives (in the momentum oper)

# The Klein-Gordon Equation

- So much for the Schrödinger equation. But maybe we can do a quick fix? Let's go back to this equation:

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} |\psi, t\rangle$$

but instead of expanding, we'll square the operator.

- This is not mathematically valid. But maybe by making this slight shift of axiom, the consequences won't be too severe.

The result is the Klein-Gordon Equation

$$-\hbar^2 \frac{\partial^2}{\partial t^2} |\psi, t\rangle = (\mathbf{p}^2 c^2 + m^2 c^4) |\psi, t\rangle$$

# The Klein-Gordon Equation

$$-\hbar^2 \frac{\partial^2}{\partial t^2} |\psi, t\rangle = (\mathbf{p}^2 c^2 + m^2 c^4) |\psi, t\rangle$$

- Is this consistent with relativity?

Yes. Two observers will agree on this equation.

- Is this consistent with quantum mechanics?

No. It doesn't follow the Schrödinger Equation. Moreover, the norm of a state is not constant (so probability is not conserved).

I'll prove both of these claims in the attached document.

# The Dirac Equation

- Let's make another attempt, this time just for spin-1/2 particles

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} -i\hbar c(\alpha^j)_{11}\partial_j + mc^2(\beta)_{11} & -i\hbar c(\alpha^j)_{12}\partial_j + mc^2(\beta)_{12} \\ -i\hbar c(\alpha^j)_{21}\partial_j + mc^2(\beta)_{21} & -i\hbar c(\alpha^j)_{22}\partial_j + mc^2(\beta)_{22} \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

- Is this consistent with QM?

Yes. It's just the Schrödinger Equation with a particular form of the Hamiltonian.

- Is this consistent with relativity?

Yes, provided that the following are true:

$$\begin{aligned} \{\alpha^j, \alpha^k\}_{ab} &= 2\delta^{jk}\delta_{ab} \\ \{\alpha^j, \beta\}_{ab} &= 0 \\ (\beta^2)_{ab} &= \delta_{ab} \end{aligned}$$

Demonstrating this is outside the scope of this section.



# The Dirac Equation

- We've done it! An equation that is consistent with relativity and quantum mechanics.
- But, some problems.
  - The matrices should be  $2 \times 2$ , since they represent the two electron spins. But there are not four  $2 \times 2$  matrices that obey the anti-commutation relations!  $3 \times 3$  is no good either, since the dimensionality must be even (see problem 1.1 for proofs of these statements). So, must have  $4 \times 4$  (or higher) dimensionality. What are the other two dimensions, physically?
  - Acting on a momentum eigenstate, the eigenvalues are  $E(p)$ ,  $E(p)$ ,  $-E(p)$ , and  $-E(p)$ . There's no ground state – any particle could emit photons to achieve successively lower energy states.
    - Dirac postulated hole theory.
  - Haven't dealt with bosons.

# The Dirac Equation

- So, two options:

1. Dirac Equation is wrong: quantum mechanics and relativity can't be combined. There is instead some "super-theory" which reduces to QM and Relativity in certain limits.
2. Dirac Equation is right, but has certain subtleties that need to be carefully considered.

- It turns out that the Dirac Equation is right. Let's write it here in a more compact way:

$$(c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta mc^2) \psi = i\hbar \frac{\partial \psi}{\partial t}$$

In this case  $\mathbf{p}$  and  $\boldsymbol{\alpha}$  are 3-vectors; the three components of  $\boldsymbol{\alpha}$ , and also  $\beta$  are 4x4 vectors.

- We won't be able to successfully interpret the Dirac Equation for a long time.

# Time & Space

- The problem with the Dirac Equation – and indeed with quantum mechanics – is this:
  - Position is an operator
  - Time is a scalar label. It is not the eigenvalue of any operator.

With time and space treated asymmetrically from the start, it is unsurprising that we have difficulty combining them into one equation!

- Two solutions:
  - Make position a label. This is what we do in QFT by definition.
  - Make time an operator. This works perfectly well, but it is difficult (since there is an infinite redundancy in any definition of proper time). However, it suggests certain generalizations that are useful, for example in string theory.

# Time & Space

- In QFT, time and position are both simply labels.
  - Labels on what? On operators!
  - Assign an operator to each point in spacetime – call that operator  $\varphi(\mathbf{x}, t)$ , with  $\mathbf{x}$  and  $t$  just labeling which point is being referred to.
  
- Our task now is twofold:
  - 1) Write ordinary quantum mechanics as a quantum field theory.
  - 2) Make it relativistic.

# QFT of QM

- Take a quantum field  $a(\mathbf{x})$ , defined by the commutation relations

$$\begin{aligned} [a(\mathbf{x}), a(\mathbf{x}')] &= 0 \\ [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] &= 0 \\ [a^\dagger(\mathbf{x}), a(\mathbf{x}')] &= \delta^3(\mathbf{x} - \mathbf{x}') \end{aligned}$$

- Now these fields will obey the Schrödinger Equation, where we have

$$H = \int d^3x a^\dagger(\mathbf{x}) \left( -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) + \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})$$

$$|\psi, t\rangle = \int d^3x_1 \dots d^3x_n \psi(x_1, \dots, x_n; t) a^\dagger(x_1) \dots a^\dagger(x_n) |0\rangle$$

# QFT of QM

- We will show in the problem 1.2 that if and only if this Schrödinger Equation is satisfied, then we have also satisfied the QM analog:

$$i\hbar \frac{\partial}{\partial t} \psi = \left[ \sum_{j=1}^n \left( -\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) + \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \right] \psi$$

- It is easy to interpret  $a$  and  $a^\dagger$  as annihilation and creation operators respectively, and  $|0\rangle$  as the vacuum. Hence,  
 $a|0\rangle = 0$ .

- We can also define a number operator.  $N = \int d^3x a^\dagger(\mathbf{x})a(\mathbf{x})$   
This commutes with the Hamiltonian, (prbm. 1.3) and so it is conserved. But, terms could be added to the Hamiltonian that don't commute with  $N$ , and so number can change for processes with certain Hamiltonians (like decay processes).

# QFT of QM

- I will show later (easy to show) that only the symmetric part of  $\Psi$  will survive the integral in our definition of  $\Psi$ . Hence, we're so far discussing bosons, where the wave function is symmetric under particle interchange.

- For fermions, our relations are instead:

$$\begin{aligned}\{a(\mathbf{x}), a(\mathbf{x}')\} &= 0 \\ \{a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')\} &= 0 \\ \{a^\dagger(\mathbf{x}), a(\mathbf{x}')\} &= \delta^3(\mathbf{x} - \mathbf{x}')\end{aligned}$$

And the wave function is antisymmetric under particle interchange

- No way to get Boltzmann statistics (which assume that quantum effects are negligible).
  - So, our course will deal with elementary particles directly, not with arbitrary fields made up of indeterminate particles.

# What's next

- Next time:
  - Discuss Lorentz Invariance so that we know how to impose relativity on our QFT.
- Next few months:
  - Make our field theory compatible with relativity,
  - Describe spin-zero particles as completely and correctly as possible
- By the end of the year:
  - Generalize to spin-1/2 and spin-1
  - Construct the Standard Model of Particle Physics