Muon & Tau Lifetime

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Abstract

We derive the expected lifetime of the muon, assuming only the Feynman Rules, Fermi's Golden Rule, the Completeness Relations (for Dirac Spinors), and the definition of the γ matrices (anti-commutator). Negligible masses or momenta are dropped. We then this result to estimate the tau lifetime, making an additional assumption of the CKM matrix.

1 Requisite Math

The γ matrices are defined with the following anticommutation relation: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where we take the Minkowski metric to be $\eta^{\mu\nu} = diag(1, -1, -1, -1)$. We claim:

1. $(\gamma^5)^2 = 1$.
   Proof.
   From the anticommutation relations, $2\gamma^\mu\gamma^\mu = 2\eta^{\mu\mu}$. Also,
   $\gamma^5\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$. Anticommuting,
   $\gamma^5\gamma^5 = -\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3$. Now $\gamma^\mu\gamma^\mu = \eta^{\mu\mu}$ (proven above), so:
   $\gamma^5\gamma^5 = +1$.

2. The trace of an odd number of γ matrices is zero.
   Proof.
   As shown above, $Tr(\gamma^{i_1}...\gamma^{i_n}) = Tr(\gamma^5\gamma^5\gamma^{i_1}...\gamma^{i_n})$, where n is odd.
   Using the cyclic property of the trace, $Tr(\gamma^{i_1}...\gamma^{i_n}) = Tr(\gamma^5\gamma^{i_1}...\gamma^{i_n}\gamma^5)$
   Now, using the definition of γ^5 and the anticommutation relation:
   $$\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5$$
   Hence, anticommuting γ^5 through an odd number of γ matrices gives:
   $$Tr(\gamma^{i_1}...\gamma^{i_n}) = -Tr(\gamma^5\gamma^5\gamma^{i_1}...\gamma^{i_n}) = -Tr(\gamma^{i_1}...\gamma^{i_n})$$
   Hence, $Tr(\gamma^{i_1}...\gamma^{i_n}) = 0$.

3. $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$.
   Proof.
   By cyclic property of trace, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)$
   Anticommutating, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\eta^{\sigma\mu}Tr(\gamma^\nu\gamma^\rho) - Tr(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma)$
   Anticommutating, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\eta^{\sigma\mu}Tr(\gamma^\nu\gamma^\rho) - 2\eta^{\rho\mu}Tr(\gamma^\nu\gamma^\sigma) + Tr(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$
   Anticommutating, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\eta^{\sigma\mu}Tr(\gamma^\nu\gamma^\rho) - 2\eta^{\rho\mu}Tr(\gamma^\nu\gamma^\sigma) + 2\eta^{\nu\mu}Tr(\gamma^\rho\gamma^\sigma) - Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$
   Simplifying: $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \eta^{\sigma\mu}Tr(\gamma^\nu\gamma^\rho) - \eta^{\rho\mu}Tr(\gamma^\nu\gamma^\sigma) + \eta^{\nu\mu}Tr(\gamma^\rho\gamma^\sigma)$
   Now notice that $Tr(\gamma^\rho\gamma^\sigma) = \frac{1}{2}Tr(\gamma^\rho\gamma^\sigma) + \frac{1}{2}Tr(\gamma^\rho\gamma^\sigma) = \frac{1}{2}Tr(\{\gamma^\rho, \gamma^\sigma\}) = 4\eta^{\rho\sigma}$
   Hence, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

4. $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 4i\varepsilon^{\mu\nu\rho\sigma}$.
   Proof.
   First, note that it follows from the anticommutation relations that $\gamma^i\gamma^i = 1$.
   Then, $Tr(\gamma^\mu\gamma^\nu\gamma^5) = Tr(\gamma^i\gamma^i\gamma^\mu\gamma^\nu\gamma^5) = Tr(\gamma^i\gamma^\mu\gamma^\nu\gamma^5\gamma^i)$, where $i \neq \mu$ and $i \neq \nu$
   We showed before that γ^5 anticommutes with everything, and γ^i will clearly commute with any gamma matrix to

which it is not equal. Hence, we anticommute three times, proving that $Tr(\gamma^\mu\gamma^\nu\gamma^5) = 0$.

Now consider $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)$. If any of these are equal, then we can commute to recover the situation above, and the answer is zero. If all four gammas appear, then the sign of our answer must change if two indices are swapped, due to anticommutation relations. Hence, $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = k\varepsilon^{\mu\nu\rho\sigma}$, where k is a proportionality constant.

To determine the proportionality constant, consider $Tr(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5) = k\varepsilon^{0123} = k\eta^{00}\eta^{11}\eta^{22}\eta^{33}\varepsilon_{0123} = -k$. Further, $Tr(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5) = -iTr(\gamma^5\gamma^5) = -iTr(I) = -4i$. Thus, $-k = -4i \implies k = 4i$, which proves the claim.

Further, we claim that if we have a symmetric matrix S and antisymmetric matrices A, then $M^{\mu\nu}M_{\mu\nu} = (S^{\mu\nu}+A^{\mu\nu})(S_{\mu\nu}+A_{\mu\nu}) = S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu}$.

Proof.

$$M^{\mu\nu}M_{\mu\nu} = S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu} + S^{\mu\nu}A_{\mu\nu} + A^{\mu\nu}S_{\mu\nu}$$

Now μ and ν are dummy indices, so let's switch them:

$$M^{\mu\nu}M_{\mu\nu} = S^{\nu\mu}S_{\nu\mu} + A^{\nu\mu}A_{\nu\mu} + S^{\nu\mu}A_{\nu\mu} + A^{\nu\mu}S_{\nu\mu}$$

Now we can switch back: S can switch "free of charge"; A can switch, at the "cost" of a minus sign. Then,

$$M^{\mu\nu}M_{\mu\nu} = S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu} - S^{\mu\nu}A_{\mu\nu} - A^{\mu\nu}S_{\mu\nu}$$

Equating two of these results, we have proven that:

$$S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu} - S^{\mu\nu}A_{\mu\nu} - A^{\mu\nu}S_{\mu\nu} = M^{\mu\nu}M_{\mu\nu} = S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu} + S^{\mu\nu}A_{\mu\nu} + A^{\mu\nu}S_{\mu\nu}$$

This implies:

$$-(S^{\mu\nu}A_{\mu\nu} + A^{\mu\nu}S_{\mu\nu}) = S^{\mu\nu}A_{\mu\nu} + A^{\mu\nu}S_{\mu\nu}$$

Hence,

$$S^{\mu\nu}A_{\mu\nu} + A^{\mu\nu}S_{\mu\nu} = 0$$

which implies:

$$M^{\mu\nu}M_{\mu\nu} = S^{\mu\nu}S_{\mu\nu} + A^{\mu\nu}A_{\mu\nu}$$

Finally, we claim that $\varepsilon^{abcd}\varepsilon_{ebfd} = -2\delta_{ae}\delta_{cf} + 2\delta_{af}\delta_{ce}$.

Proof.

Using the definition of the Levi-Cevita Tensor, we can rewrite this as: $\varepsilon^{bdac}\varepsilon_{bdef}$.

If a = c or e = f, then we are done.

There is a sum over b, d. The only terms that will contribute are the terms in which b, d choose the two values that have not already been claimed by the other indices.

Hence, the contributing terms have either a = e and c = f, or a = f and c = e. That is:

$$\varepsilon^{bdac}\varepsilon_{bdef} = \varepsilon^{bdac}\varepsilon_{bdac}\delta_{ae}\delta_{cf} + \varepsilon^{bdac}\varepsilon_{bdca}\delta_{af}\delta_{ce}$$

Reversing the last two indices in the last term (at the cost of a minus sign) gives:

$$\varepsilon^{bdac}\varepsilon_{bdef} = \varepsilon^{bdac}\varepsilon_{bdac}\delta_{ae}\delta_{cf} - \varepsilon^{bdac}\varepsilon_{bdac}\delta_{af}\delta_{ce}$$

The Levi-Cevita tensors are the same, but the superscript is different from the subscript by a factor of $\eta^{00}\eta^{11}\eta^{22}\eta^{33} = -1$

for contributing terms. Hence, the product of the Levi Cevita Tensors gives -1. There are two contributing terms, so:

$$\varepsilon^{bdac}\varepsilon_{bdef} = -2\delta_{ae}\delta_{cf} + 2\delta_{af}\delta_{ce}$$

2 Muon Decay Lifetime Calculation

The decay process is $\mu \to e + \overline{\nu}_e + \nu_\mu$.

We'll do our labeling with subscripts. We'll define the muon as particle 1, the muon neutrino as particle 2, the electron neutrino as particle 3, and the electron as particle 4.

Let's construct $\mathscr{M}$ using the Feynman Rules.

1. i, since the presented Feynman rules are for $-i\mathscr{M}$.
2. The W^- propagator:
$$-\frac{i(g_{\mu\nu} - p_\mu p_\nu)}{p^2 - M_W^2}$$
Since M_W is so large, we can simplify to:
$$\frac{ig_{\mu\nu}}{M_W^2}$$
3. Vertex 1. One particle (the muon, k_1) is going toward the vertex, so it is represented with u; the other particle (the ν_μ, k_2) is going away from the vertex, so it is represented with $\overline{u}$. The vertex factor is:
$$\overline{u}(k_2, s_2)\left(-i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1}{2}(1-\gamma^5)\right)u(k_1, s_1)$$
4. Vertex 2. One particle (the electron, k_4) is going away from the vertex, so it is represented with $\overline{u}$; the antiparticle (the $\overline{\nu}_e$, k_3) is also going away from the vertex, so it is represented with v (the convention for u and v is opposite). The vertex factor is:
$$\overline{u}(k_4, s_4)\left(-i\frac{g}{\sqrt{2}}\gamma^\nu \frac{1}{2}(1-\gamma^5)\right)v(k_3, s_3)$$
5. The external lines contribute nothing beyond the spinors already included.

Multiplying all these together, we are left with:

$$\mathscr{M} = i\frac{ig_{\mu\nu}}{M_W^2}\overline{u}(k_2, s_2)\left(-i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1}{2}(1-\gamma^5)\right)u(k_1, s_1)\overline{u}(k_4, s_4)\left(-i\frac{g}{\sqrt{2}}\gamma^\nu \frac{1}{2}(1-\gamma^5)\right)v(k_3, s_3)$$

Our four factors of i and two negative signs all cancel:

$$\mathscr{M} = \frac{g_{\mu\nu}}{M_W^2}\overline{u}(k_2, s_2)\left(\frac{g}{\sqrt{2}}\gamma^\mu \frac{1}{2}(1-\gamma^5)\right)u(k_1, s_1)\overline{u}(k_4, s_4)\left(\frac{g}{\sqrt{2}}\gamma^\nu \frac{1}{2}(1-\gamma^5)\right)v(k_3, s_3)$$

Bringing the constants to the front gives:

$$\mathscr{M} = \frac{g^2}{8M_W^2}g_{\mu\nu}\overline{u}(k_2, s_2)\gamma^\mu(1-\gamma^5)u(k_1, s_1)\overline{u}(k_4, s_4)\gamma^\nu(1-\gamma^5)v(k_3, s_3)$$

Let's take the usual definition of G:

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

This gives:

$$\mathscr{M} = \frac{G}{\sqrt{2}}g_{\mu\nu}\overline{u}(k_2, s_2)\gamma^\mu(1-\gamma^5)u(k_1, s_1)\overline{u}(k_4, s_4)\gamma^\nu(1-\gamma^5)v(k_3, s_3)$$

We'll use the metric to lower the second γ^ν:

$$\mathscr{M} = \frac{G}{\sqrt{2}}\overline{u}(k_2, s_2)\gamma^\mu(1-\gamma^5)u(k_1, s_1)\overline{u}(k_4, s_4)\gamma_\mu(1-\gamma^5)v(k_3, s_3)$$

Finally, we'll temporarily suppress the spin indices:

$$\mathscr{M} = \frac{G}{\sqrt{2}}\overline{u}(k_2)\gamma^\mu(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_\mu(1-\gamma^5)v(k_3)$$

The Hermitian conjugate of this is given by $\mathscr{M}^\dagger$:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \left(\overline{u}(k_2)\gamma^\mu(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_\mu(1-\gamma^5)v(k_3)\right)^\dagger$$

Distributing the Hermitian conjugate gives:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} v(k_3)^\dagger(1-\gamma^5)^\dagger\gamma_\mu^\dagger\overline{u}(k_4)^\dagger u(k_1)^\dagger(1-\gamma^5)^\dagger\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

$1^\dagger = 1$ and $\gamma^{5\dagger} = \gamma^5$, so:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} v(k_3)^\dagger(1-\gamma^5)\gamma_\mu^\dagger\overline{u}(k_4)^\dagger u(k_1)^\dagger(1-\gamma^5)\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

$\gamma^0\gamma^0 = 1$, so let's insert a few γ^0s.

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} v(k_3)^\dagger\gamma^0\gamma^0(1-\gamma^5)\gamma_\mu^\dagger\overline{u}(k_4)^\dagger u(k_1)^\dagger\gamma^0\gamma^0(1-\gamma^5)\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

$v^\dagger\gamma^0 = \overline{v}$. Hence,

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)\gamma^0(1-\gamma^5)\gamma_\mu^\dagger\overline{u}(k_4)^\dagger\overline{u}(k_1)\gamma^0(1-\gamma^5)\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

Distributing the remaining γ^0s:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(\gamma^0-\gamma^0\gamma^5)\gamma_\mu^\dagger\overline{u}(k_4)^\dagger\overline{u}(k_1)(\gamma^0-\gamma^0\gamma^5)\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

γ^5 anticommutes with the other γs, hence:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(\gamma^0+\gamma^5\gamma^0)\gamma_\mu^\dagger\overline{u}(k_4)^\dagger\overline{u}(k_1)(\gamma^0+\gamma^5\gamma^0)\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

Again factoring out the γ^0s, we have:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(1+\gamma^5)\gamma^0\gamma_\mu^\dagger\overline{u}(k_4)^\dagger\overline{u}(k_1)(1+\gamma^5)\gamma^0\gamma^{\mu\dagger}\overline{u}(k_2)^\dagger$$

We place in two more copies $\gamma^0\gamma^0$. Thus:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(1+\gamma^5)\gamma^0\gamma_\mu^\dagger\gamma^0\gamma^0\overline{u}(k_4)^\dagger\overline{u}(k_1)(1+\gamma^5)\gamma^0\gamma^{\mu\dagger}\gamma^0\gamma^0\overline{u}(k_2)^\dagger$$

We note now that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Multiplying on the left and the right by γ^0, we find that $\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$. This leads to:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(1+\gamma^5)\gamma^0\gamma_\mu^\dagger\gamma^0\gamma^0\overline{u}(k_4)^\dagger\overline{u}(k_1)(1+\gamma^5)\gamma^\mu\gamma^0\overline{u}(k_2)^\dagger$$

The previous fact can be used again if we multiply both sides by the metric. Then,

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(1+\gamma^5)\gamma_\mu\gamma^0\overline{u}(k_4)^\dagger\overline{u}(k_1)(1+\gamma^5)\gamma^\mu\gamma^0\overline{u}(k_2)^\dagger$$

As noted before, $\overline{u} = u^\dagger\gamma^0$. We dagger both sides to find:

$$\overline{u}^\dagger = (u^\dagger\gamma^0)^\dagger = \gamma^{0\dagger}u^{\dagger\dagger} = \gamma^0 u$$

Multiplying both sides of that by γ^0 on the left gives:

$$\gamma^0\overline{u}^\dagger = \gamma^0\gamma^0 u = u$$

Finally, we use this identity in our expression for $\mathscr{M}^\dagger$ to obtain:

$$\mathscr{M}^\dagger = \frac{G}{\sqrt{2}} \overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)$$

Now we are ready to put these together and obtain our amplitude squared:

$$|\mathscr{M}|^2 = \mathscr{M}^\dagger \mathscr{M}$$

which gives:

$$|\mathscr{M}|^2 = \frac{G}{\sqrt{2}}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)\frac{G}{\sqrt{2}}\overline{u}(k_2)\gamma^\nu(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

Bringing the constants to the front:

$$|\mathscr{M}|^2 = \frac{G^2}{2}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)\overline{u}(k_2)\gamma^\nu(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

This is almost a good expression, but it remains to take care of the spins.

1. Average over initial spins (s_1). The muon in question has exactly one spin out of the the two possibilities, so we will take the average of both. This means summing over the two and then dividing by two.

2. Sum over the final spins (s_2, s_3, s_4). The particles in question can decay into a number of final states; each possibility opens up new channels, causing the particle to decay faster (ie increasing the amplitude).

Hence, we will sum over all four spins and divide by $\frac{1}{2}$.

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_1,s_2,s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)\overline{u}(k_2)\gamma^\nu(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

The middle portion of this equation is the most interesting. Let us consider that part separately:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\left[\sum_{s_1,s_2}\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)\overline{u}(k_2)\gamma^\nu(1-\gamma^5)u(k_1)\right]\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

These are essentially a bunch of matrices and spinors multiplied together, so let's write this in index notation. For convenience, we'll use Latin letters:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\left[\sum_{s_1,s_2}\overline{u}(k_1)_a\left[(1+\gamma^5)\gamma^\mu\right]_{ab}u(k_2)_b\overline{u}(k_2)_c\left[\gamma^\nu(1-\gamma^5)\right]_{cd}u(k_1)_d\right]\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

Now we don't have to worry about commutation, so we can write:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\left[\sum_{s_1,s_2}u(k_1)_d\overline{u}(k_1)_a\left[(1+\gamma^5)\gamma^\mu\right]_{ab}u(k_2)_b\overline{u}(k_2)_c\left[\gamma^\nu(1-\gamma^5)\right]_{cd}\right]\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

Note that this has the same index at the beginning and the end, which is the trace.

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\left[\sum_{s_1,s_2}Tr\left(u(k_1)\overline{u}(k_1)(1+\gamma^5)\gamma^\mu u(k_2)\overline{u}(k_2)\gamma^\nu(1-\gamma^5)\right)\right]\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

Let's use the completeness relation, $\sum_{s_1}\overline{u}(k_1,s_1)u(k_1,s_1) = \not{k}_1 + m$ and $\sum_{s_2} = \not{k}_2$. Then,

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)Tr\left[(\not{k}_1+m)(1+\gamma^5)\gamma^\mu\ \not{k}_2\gamma^\nu(1-\gamma^5)\right]\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

The trace is now irrelevant to the rest of the problem, so we'll move it in front:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4}Tr\left[(\not{k}_1+m)(1+\gamma^5)\gamma^\mu\ \not{k}_2\gamma^\nu(1-\gamma^5)\right]\sum_{s_3,s_4}\overline{v}(k_3)(1+\gamma^5)\gamma_\mu u(k_4)\overline{u}(k_4)\gamma_\nu(1-\gamma^5)v(k_3)$$

In the same way, we convert the remaining sum to a trace:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} Tr\left[(\not{k}_1 + m)(1+\gamma^5)\gamma^\mu \ \not{k}_2\gamma^\nu(1-\gamma^5)\right] Tr\left[\not{k}_3(1+\gamma^5)\gamma_\mu \ \not{k}_4\gamma_\nu(1-\gamma^5)\right]$$

γ^μ and γ^5 anti-commute, so we can rewrite this:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} Tr\left[(\not{k}_1 + m)\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5)\right] Tr\left[\not{k}_3\gamma_\mu(1-\gamma^5) \ \not{k}_4\gamma_\nu(1-\gamma^5)\right]$$

Rearranging again, we find:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5)(\not{k}_1 + m)\right] Tr\left[\gamma_\mu(1-\gamma^5) \ \not{k}_4\gamma_\nu(1-\gamma^5) \ \not{k}_3\right]$$

Interestingly enough, the mass terms vanish! To see this, let's take the first trace and distribute the first five terms into the sixth:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1 + \gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5)m)\right] Tr\left[\gamma_\mu(1-\gamma^5) \ \not{k}_4\gamma_\nu(1-\gamma^5) \ \not{k}_3\right]$$

Traces are linear, so:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} \left(Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] + Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5)m)\right]\right) Tr\left[\gamma_\mu(1-\gamma^5) \ \not{k}_4\gamma_\nu(1-\gamma^5) \ \not{k}_3\right]$$

Consider this second term. Let's do the long multiplication:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5)m)\right] = Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu m\right] - Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu m\right] - Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu\gamma^5 m)\right] + Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu\gamma^5 m)\right]$$

All these terms have an odd number of gamma matrices (recall that γ^5 has four gamma matrices inside it), so this vanishes, leaving:

$$\overline{|\mathscr{M}|^2} = \frac{G^2}{4} Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] Tr\left[\gamma_\mu(1-\gamma^5) \ \not{k}_4\gamma_\nu(1-\gamma^5) \ \not{k}_3\right]$$

Consider this first trace. Multiplying out the binomials, we have:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu \ \not{k}_1\right] - Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu \ \not{k}_1\right] - Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu\gamma^5 \ \not{k}_1\right] + Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu\gamma^5 \ \not{k}_1\right]$$

Remembering that $\not{a} = a_\alpha\gamma^\alpha$, we can anticommute the γ^5 terms in the third and fourth terms, obtaining:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu \ \not{k}_1\right] - Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu \ \not{k}_1\right] - Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu \ \not{k}_1\right] + Tr\left[\gamma^\mu\gamma^5\gamma^5 \ \not{k}_2\gamma^\nu \ \not{k}_1\right]$$

Since $(\gamma^5)^2 = 1$, two of these terms combine:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 2Tr\left[\gamma^\mu \ \not{k}_2\gamma^\nu \ \not{k}_1\right] - 2Tr\left[\gamma^\mu\gamma^5 \ \not{k}_2\gamma^\nu \ \not{k}_1\right]$$

Using the cyclic property of the trace, and also the definition of $\not{a}$, we have:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 2k_{1\alpha}k_{2\beta}Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] - 2k_{1\alpha}k_{2\beta}Tr\left[\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^5\right]$$

Now we can use the identities we determined previously:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 8k_{1\alpha}k_{2\beta}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu}) - 8k_{1\alpha}k_{2\beta}i\varepsilon^{\beta\nu\alpha\mu}$$

which is:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 8k_{1\alpha}k_{2\beta}\eta^{\mu\alpha}\eta^{\nu\beta} - 8k_{1\alpha}k_{2\beta}\eta^{\mu\nu}\eta^{\alpha\beta} + 8k_{1\alpha}k_{2\beta}\eta^{\mu\beta}\eta^{\alpha\nu} - 8k_{1\alpha}k_{2\beta}i\varepsilon^{\beta\nu\alpha\mu}$$

Since the metric is diagonal:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 8k_{1\mu}k_{2\nu}\eta^{\mu\mu}\eta^{\nu\nu} - 8k_{1\alpha}k_{2\beta}\eta^{\mu\nu}\eta^{\alpha\beta} + 8k_{1\nu}k_{2\mu}\eta^{\mu\mu}\eta^{\nu\nu} - 8k_{1\alpha}k_{2\beta}i\varepsilon^{\beta\nu\alpha\mu}$$

Simplifying:

$$Tr\left[\gamma^\mu(1-\gamma^5) \ \not{k}_2\gamma^\nu(1-\gamma^5) \ \not{k}_1\right] = 8k_1^\mu k_2^\nu - 8(k_1\cdot k_2)\eta^{\mu\nu} + 8k_1^\nu k_2^\mu - 8k_{1\alpha}k_{2\beta}i\varepsilon^{\beta\nu\alpha\mu}$$

Now we want to consider

$$Tr\left[\gamma^\mu(1-\gamma^5)\ \not{k}_2\gamma^\nu(1-\gamma^5)\ \not{k}_1\right]Tr\left[\gamma_\mu(1-\gamma^5)\ \not{k}_4\gamma_\nu(1-\gamma^5)\ \not{k}_3\right]=$$

$$\left(8k_1^\mu k_2^\nu-8(k_1\cdot k_2)\eta^{\mu\nu}+8k_1^\nu k_2^\mu-8k_{1\alpha}k_{2\beta}i\varepsilon^{\beta\nu\alpha\mu}\right)\left(8k_{3\mu}k_{4\nu}-8(k_3\cdot k_4)\eta_{\mu\nu}+8k_{3\nu}k_{4\mu}-8k_3^\gamma k_4^\delta i\varepsilon_{\delta\nu\gamma\mu}\right)$$

In each of these, the first three terms are even and the last is odd. As we showed in previously, we can multiply these separately. Hence,

$$Tr\left[\gamma^\mu(1-\gamma^5)\ \not{k}_2\gamma^\nu(1-\gamma^5)\ \not{k}_1\right]Tr\left[\gamma_\mu(1-\gamma^5)\ \not{k}_4\gamma_\nu(1-\gamma^5)\ \not{k}_3\right]=$$

$$(8k_1^\mu k_2^\nu-8(k_1\cdot k_2)\eta^{\mu\nu}+8k_1^\nu k_2^\mu)(8k_{3\mu}k_{4\nu}-8(k_3\cdot k_4)\eta_{\mu\nu}+8k_{3\nu}k_{4\mu})-64k_{1\alpha}k_{2\beta}\varepsilon^{\beta\nu\alpha\mu}k_3^\gamma k_4^\delta\varepsilon_{\delta\nu\gamma\mu}$$

Multiplying both sides by the necessary constants, we can simplify this as:

$$\overline{|\mathscr{M}|^2}=\frac{G^2}{4}\left[(8k_1^\mu k_2^\nu-8(k_1\cdot k_2)\eta^{\mu\nu}+8k_1^\nu k_2^\mu)(8k_{3\mu}k_{4\nu}-8(k_3\cdot k_4)\eta_{\mu\nu}+8k_{3\nu}k_{4\mu})-64k_{1\alpha}k_{2\beta}k_3^\gamma k_4^\delta\varepsilon^{\beta\nu\alpha\mu}\varepsilon_{\delta\nu\gamma\mu}\right]$$

Simplifying,

$$\overline{|\mathscr{M}|^2}=16G^2\left[(k_1^\mu k_2^\nu-(k_1\cdot k_2)\eta^{\mu\nu}+k_1^\nu k_2^\mu)(k_{3\mu}k_{4\nu}-(k_3\cdot k_4)\eta_{\mu\nu}+k_{3\nu}k_{4\mu})-k_{1\alpha}k_{2\beta}k_3^\gamma k_4^\delta\varepsilon^{\beta\nu\alpha\mu}\varepsilon_{\delta\nu\gamma\mu}\right]$$

Doing the long multiplication, the terms in parenthesis become:

$$(k_1\cdot k_3)(k_2\cdot k_4)-(k_1\cdot k_2)(k_3\cdot k_4)+(k_1\cdot k_4)(k_2\cdot k_3)-(k_1\cdot k_2)(k_3\cdot k_4)+4(k_1\cdot k_2)(k_3\cdot k_4)$$

$$-(k_1\cdot k_2)(k_3\cdot k_4)+(k_1\cdot k_3)(k_2\cdot k_4)-(k_1\cdot k_2)(k_3\cdot k_4)+(k_1\cdot k_4)(k_2\cdot k_3)$$

Combining similar terms, this reduces, giving:

$$\overline{|\mathscr{M}|^2}=16G^2\left[2(k_1\cdot k_3)(k_2\cdot k_4)+2(k_1\cdot k_4)(k_2\cdot k_3)-k_{1\alpha}k_{2\beta}k_3^\gamma k_4^\delta\varepsilon^{\beta\nu\alpha\mu}\varepsilon_{\delta\nu\gamma\mu}\right]$$

The Levi-Cevita tensor expands as shown above, giving:

$$\overline{|\mathscr{M}|^2}=16G^2\left[2(k_1\cdot k_3)(k_2\cdot k_4)+2(k_1\cdot k_4)(k_2\cdot k_3)+2k_{1\alpha}k_{2\beta}k_3^\gamma k_4^\delta\delta_{\beta\delta}\delta_{\alpha\gamma}-2k_{1\alpha}k_{2\beta}k_3^\gamma k_4^\delta\delta_{\beta\gamma}\delta_{\alpha\delta}\right]$$

Hence,

$$\overline{|\mathscr{M}|^2}=16G^2\left[2(k_1\cdot k_3)(k_2\cdot k_4)+2(k_1\cdot k_4)(k_2\cdot k_3)+2(k_1\cdot k_3)(k_2\cdot k_4)-2(k_1\cdot k_4)(k_2\cdot k_3)\right]$$

Some terms cancel, leaving:

$$\overline{|\mathscr{M}|^2}=64G^2(k_1\cdot k_3)(k_2\cdot k_4)$$

Now we are ready to consider the infinitesimal decay rate. By definition,

$$d\Gamma=\overline{|\mathscr{M}|^2}dLIPS$$

Plugging in both the invariant amplitude and the expression for the phase space, we have:

$$d\Gamma=\frac{1}{2m}\left(64G^2(k_1\cdot k_3)(k_2\cdot k_4)\right)\frac{d^3k_2}{(2\pi)^3 2E_{k2}}\frac{d^3k_3}{(2\pi)^3 2E_{k3}}\frac{d^3k_4}{(2\pi)^3 2E_{k4}}(2\pi)^4\delta^4(k_1-k_2-k_3-k_4)$$

Bringing the constants to the front,

$$d\Gamma=\frac{G^2}{8m\pi^5}\left((k_1\cdot k_3)(k_2\cdot k_4)\right)\frac{d^3k_2}{E_{k2}}\frac{d^3k_3}{E_{k3}}\frac{d^3k_4}{E_{k4}}\delta^4(k_1-k_2-k_3-k_4)$$

Let's break apart the delta function into energy and momentum components. We work in the center of mass frame, where the muon's energy is its mass. The remaining particles have essentially no mass, so their energy is the opposite of their momentum.

$$d\Gamma=\frac{G^2}{8m\pi^5}\left((k_1\cdot k_3)(k_2\cdot k_4)\right)\frac{d^3k_2}{|\vec{k}_2|}\frac{d^3k_3}{|\vec{k}_3|}\frac{d^3k_4}{|\vec{k}_4|}\delta(m-|\vec{k}_2|-|\vec{k}_3|-|\vec{k}_4)|\delta^3(\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

Note that in the muon rest frame, $k_1=(m,0,0,0)$. Then, $k_1\cdot k_3=mE_3$, and

$$d\Gamma=\frac{G^2}{8m\pi^5}\left((k_2\cdot k_4)mE_3\right)\frac{d^3k_2}{|\vec{k}_2|}\frac{d^3k_3}{|\vec{k}_3|}\frac{d^3k_4}{|\vec{k}_4|}\delta(m-|\vec{k}_2|-|\vec{k}_3|-|\vec{k}_4)|\delta^3(\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

For the remaining dot product, consider:

$$(k_2 + k_4)^2 = k_2^2 + 2k_2 \cdot k_4 + k_4^2$$

Since we've neglected many masses, the four-vector magnitudes are zero:

$$(k_2 + k_4)^2 = 2k_2 \cdot k_4$$

Remember that $k_1 = k_2 + k_3 + k_4$, so:

$$(k_2 + k_4)^2 = 2k_2 \cdot k_4 = (k_1 - k_3)^2 = k_1^2 - 2k_1 \cdot k_3 + k_3^2$$

The muon is at rest in this frame, and the everything else has a four-magnitude of zero, so:

$$2k_2 \cdot k_4 = m^2 - 2mE_3$$

Hence,

$$d\Gamma = \frac{G^2}{16m\pi^5}\left((m^2 - 2mE_3)mE_3\right)\frac{d^3k_2}{|\vec{k}_2|}\frac{d^3k_3}{|\vec{k}_3|}\frac{d^3k_4}{|\vec{k}_4|}\delta(m - |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)|\delta^3(\vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

We clean up and also note that $E_3 = |\vec{k}_3|$, so:

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{16\pi^5}\left(m - 2|\vec{k}_3|\right)\frac{d^3k_2}{|\vec{k}_2|}\frac{d^3k_3}{|\vec{k}_3|}\frac{d^3k_4}{|\vec{k}_4|}\delta(m - |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)|\delta^3(\vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

It remains to do nine integrals. We'll start with the k_2 integral, using the delta function:

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{16\pi^5}\left(m - 2|\vec{k}_3|\right)\frac{d^3k_3 d^3k_4}{|\vec{k}_3 + \vec{k}_4||\vec{k}_3||\vec{k}_4|}\delta(m - |\vec{k}_3 + \vec{k}_4| - |\vec{k}_3| - |\vec{k}_4)|$$

It remains to do six integrals. Let's switch k_3 to polar coordinates:

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{16\pi^5}\left(m - 2|\vec{k}_3|\right)\frac{|\vec{k}_3|^2 sin(\theta) d|\vec{k}_3| d\theta d\phi d^3k_4}{|\vec{k}_3 + \vec{k}_4||\vec{k}_3||\vec{k}_4|}\delta(m - |\vec{k}_3 + \vec{k}_4| - |\vec{k}_3| - |\vec{k}_4)|$$

We can rewrite this as:

$$d\Gamma = \frac{mG^2|\vec{k}_3|^2}{16\pi^5}\left(m - 2|\vec{k}_3|\right)\frac{sin(\theta) d|\vec{k}_3| d\theta d\phi d^3k_4}{|\vec{k}_3 + \vec{k}_4||\vec{k}_4|}\delta(m - |\vec{k}_3 + \vec{k}_4| - |\vec{k}_3| - |\vec{k}_4)|$$

By the so-called "parallelogram addition rule" (a manifestation of the Law of Cosines), we can write:

$$|\vec{A} + \vec{B}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + 2|A||B|cos(\theta)$$

Hence, with the axis defined along the k_4 particle's momentum, we write:

$$d\Gamma = \frac{mG^2|\vec{k}_3|^2}{16\pi^5}\left(m - 2|\vec{k}_3|\right)\frac{sin(\theta) d|\vec{k}_3| d\theta d\phi d^3k_4}{\left(|\vec{k}_3|^2 + |\vec{k}_4|^2 + 2|\vec{k}_3||\vec{k}_4|cos(\theta)\right)|\vec{k}_4|}\delta(m - |\vec{k}_3 + \vec{k}_4| - |\vec{k}_3| - |\vec{k}_4)|$$

We now perform the ϕ integral:

$$d\Gamma = \frac{mG^2|\vec{k}_3|^2}{8\pi^4}\left(m - 2|\vec{k}_3|\right)\frac{sin(\theta) d|\vec{k}_3| d\theta d^3k_4}{\left(|\vec{k}_3|^2 + |\vec{k}_4|^2 + 2|\vec{k}_3||\vec{k}_4|cos(\theta)\right)|\vec{k}_4|}\delta(m - |\vec{k}_3 + \vec{k}_4| - |\vec{k}_3| - |\vec{k}_4)|$$

Five integrals left. Let's switch variables:

$$u^2 = |\vec{k}_3|^2 + |\vec{k}_4|^2 + 2|\vec{k}_3||\vec{k}_4|cos(\theta)$$

$$2udu = -2|\vec{k}_3||\vec{k}_4|sin(\theta)d\theta$$

Then,

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{8\pi^4}\left(m - 2|\vec{k}_3|\right)\frac{dud|\vec{k}_3|d^3k_4}{|\vec{k}_4|^2}\delta(m - u^2 - |\vec{k}_3| - |\vec{k}_4)|$$

We now perform the u integral:

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{8\pi^4}\left(m - 2|\vec{k}_3|\right)\frac{d|\vec{k}_3|d^3k_4}{|\vec{k}_4|^2}\int du\delta(m - u^2 - |\vec{k}_3| - |\vec{k}_4|)$$

What are the limits of integration? If $\theta = 0$, then $u = \sqrt{|\vec{k}_3|^2 + |\vec{k}_4|^2 + 2|\vec{k}_3||\vec{k}_4|}$, which we will call u_+. If $\theta = \pi$, then $u = \sqrt{|\vec{k}_3|^2 + |\vec{k}_4|^2 - 2|\vec{k}_3||\vec{k}_4|}$, which we will call u_-. Hence, the u integral is one if:

$$u_- < m - |\vec{k}_3| - |\vec{k}_4| < u_+$$

and zero otherwise. Factoring, we see that this condition is:

$$\pm(|\vec{k}_3| - |\vec{k}_4|) < m - |\vec{k}_3| - |\vec{k}_4| < \pm(|\vec{k}_3| + |\vec{k}_4|)$$

Taking the positive sign and solving the left inequality, we see that:

$$|\vec{k}_3| < \frac{m}{2}$$

Taking the negative sign and solving the left inequality, we see that:

$$|\vec{k}_4| < \frac{m}{2}$$

Solving the right inequality (with the positive sign), we see that:

$$|\vec{k}_3| + |\vec{k}_4| > \frac{m}{2}$$

We therefore note these constraints and take the value of the integral to be one, leaving:

$$d\Gamma = \frac{mG^2|\vec{k}_3|}{8\pi^4}\left(m - 2|\vec{k}_3|\right)\frac{d|\vec{k}_3|d^3k_4}{|\vec{k}_4|^2}$$

Rewritten a bit, we have four integrals left:

$$d\Gamma = \frac{mG^2}{8\pi^4}\left(m - 2|\vec{k}_3|\right)\frac{|\vec{k}_3|}{|\vec{k}_4|^2}d|\vec{k}_3|d^3k_4$$

Let's do the $|\vec{k}_3|$ integral next. From the constraints, we know that the upper limit is $\frac{m}{2}$ while the lower limit is $\frac{m}{2} - |\vec{k}_4|$. The integral is then:

$$d\Gamma = \frac{mG^2}{8\pi^4}\frac{1}{|\vec{k}_4|^2}d^3k_4\int |\vec{k}_3|\left(m - 2|\vec{k}_3|\right)d|\vec{k}_3|$$

We evaluate the integral between those limits, to find:

$$d\Gamma = \frac{mG^2}{8\pi^4}d^3k_4\left(\frac{m}{2} - \frac{2|\vec{k}_4|}{3}\right)$$

Three integrals left. We'll again switch to polar coordinates: this time it is trivial to evaluate the θ and ϕ components, so we have:

$$d\Gamma = \frac{mG^2}{2\pi^3}|\vec{k}_4|^2\left(\frac{m}{2} - \frac{2|\vec{k}_4|}{3}\right)d|\vec{k}_4|$$

Let's define E to be the electron energy, or $|\vec{k}_4|$. Then:

$$\frac{d\Gamma}{dE} = \frac{mG^2}{2\pi^3}E^2\left(\frac{m}{2} - \frac{2E}{3}\right)$$

Rewriting a bit:

$$\frac{d\Gamma}{dE} = \frac{m^2G^2}{4\pi^3}E^2\left(1 - \frac{4E}{3m}\right)$$

By the way, this is the energy distribution of the emitted electron. To figure out the most probable electron energy, we simply take the derivative of the right hand side and set it equal to zero. We obtain

$$\boxed{E_{prob} = \frac{m}{2}}$$

This is also the maximum energy. Note that the energy distribution at hand does not set the maximum energy of the electron; that is an external constraint set by the delta function which was required by conservation of energy. At any rate, we have:

$$\Gamma = \frac{m^2G^2}{4\pi^3}\int_0^{\frac{m}{2}} E^2\left(1 - \frac{4E}{3m}\right)dE$$

Evaluating the integral, we obtain:

$$\Gamma = \frac{m^2G^2}{4\pi^3}\frac{m^3}{48}$$

which is:

$$\Gamma = \frac{m^5G^2}{192\pi^3}$$

The muon lifetime is the inverse of this, or:

$$\boxed{\tau = \frac{192\pi^3}{m^5G^2}}$$

$G = 1.17 \times 10^{-5}$ GeV^{-2} and m = .1056584 GeV. Hence,

$$\tau = \frac{192\pi^3}{(.1056584 \text{ GeV})^5(1.17 \times 10^{-5} \text{ GeV}^{-2})^2} = 3.30 \times 10^{18}\text{GeV}^{-1}$$

To get to seconds, multiply by Planck's constant, $\hbar = 6.58 \times 10^{-25}$ Gev s.

$$\tau = 2.17\mu s$$

This differs from the experimental value by only 1.2%. The only significant source of error is our approximation that all masses are negligible except that of the μ, and that all momenta are neglible compared to the mass of the W boson.

3 Tau Lifetime

Let's consider the decay modes of the τ:

$$\tau \to \mu + \overline{\nu}_\mu + \nu_\tau$$

$$\tau \to e + \overline{\nu}_e + \nu_\tau$$

$$\tau \to d + \overline{u} + \nu_\tau \text{ (3 colors)}$$

$$\tau \to s + \overline{u} + \nu_\tau \text{ (3 colors)}$$

The τ therefore has 8 decay modes, where the μ has only one. The leptonic decay modes are equally likely (up to the mass of the lepton, which we neglected). The hadronic decay modes have a probability of $3|V_{ud}|^2 + 3|V_{us}|^2 = 2.99$. Hence, the τ is 4.99 times as likely to decay as the μ, just due to the sum over all final states.

The other issue is that the τ is much heavier (by a factor of 16.82) than the μ. Combining these effects, we have:

$$\text{lifetime}_\tau = \frac{\text{lifetime}_\mu}{4.99(16.82)^5} = 3.23 \times 10^{-13} \text{ s}$$

Where we used the lifetime of the muon calculated above. This differs from the experimental value by about 11.1%. The only significant source of error is our approximations that all masses are negligible except that of the τ, and that all momenta are neglible compared to the mass of the W boson. In fact, the τ is only 16 times heavier than the μ. Additionally, there may be a form factor in the hadronic final states.