LAGRANGIAN FORMULATION OF CLASSICAL AND PARTICLE PHYSICS

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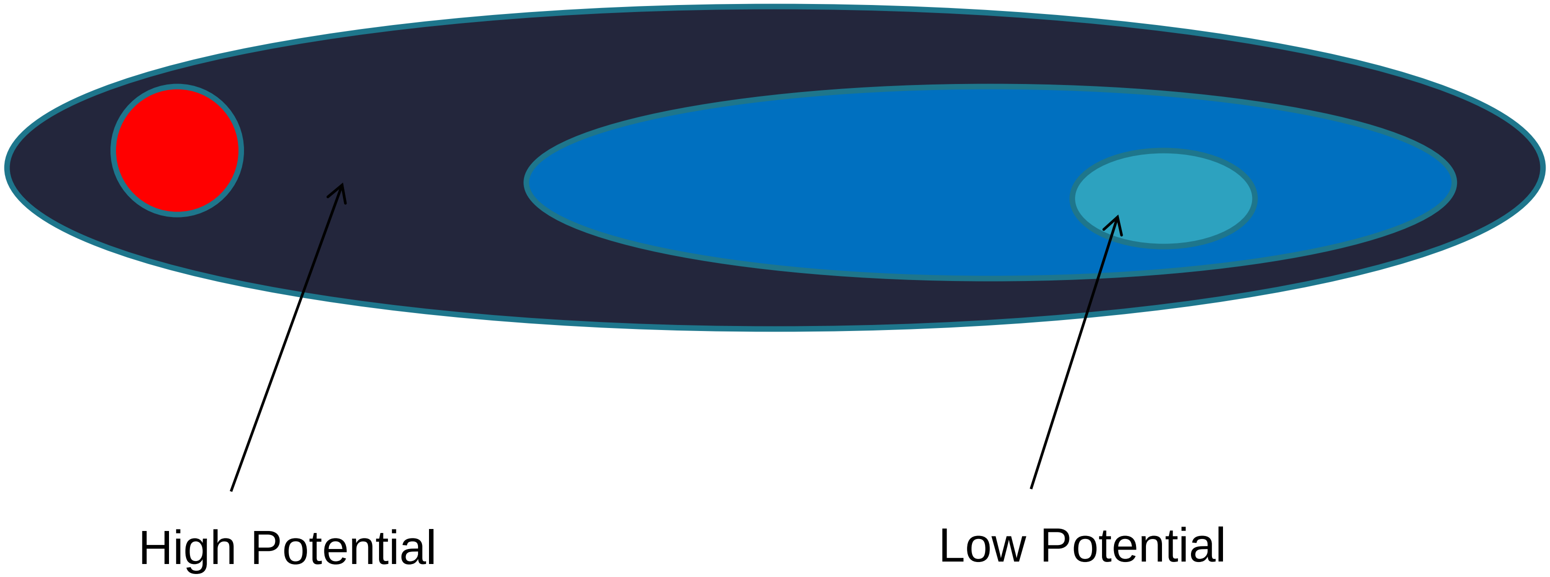
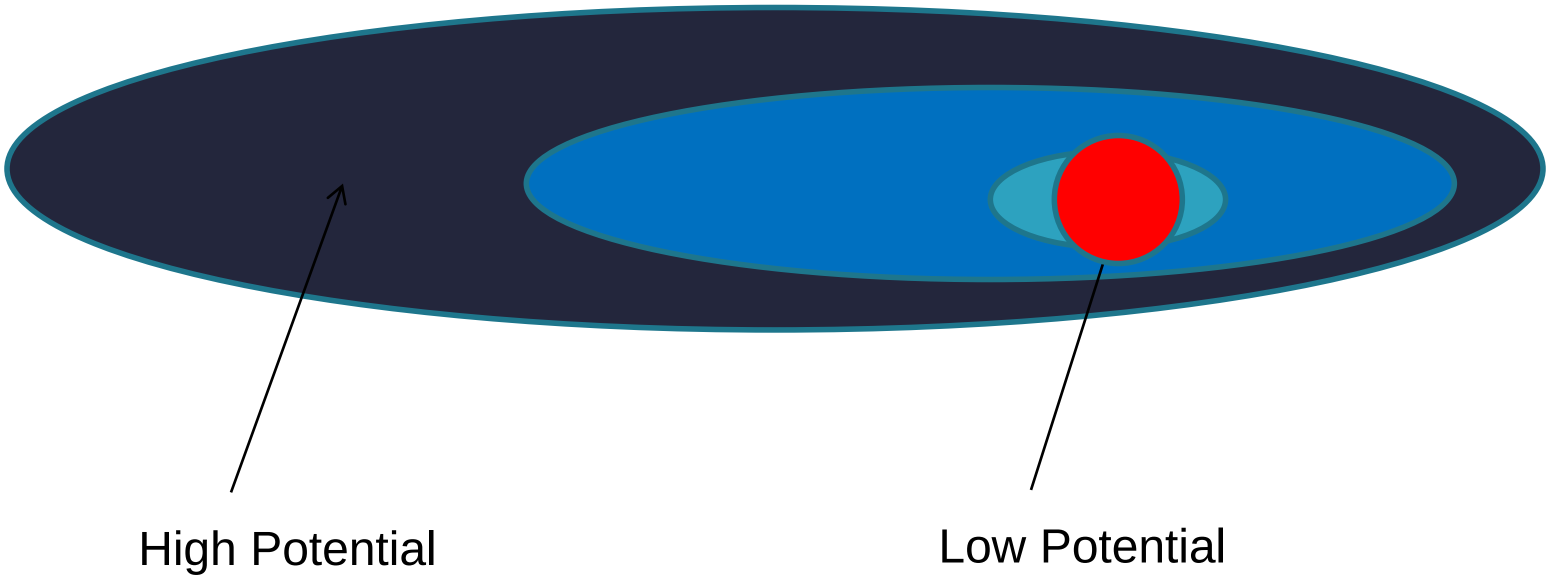
Energies & Forces

- Consider this situation, with a red ball in a blue potential field. Assume there is friction.



Energies and Forces

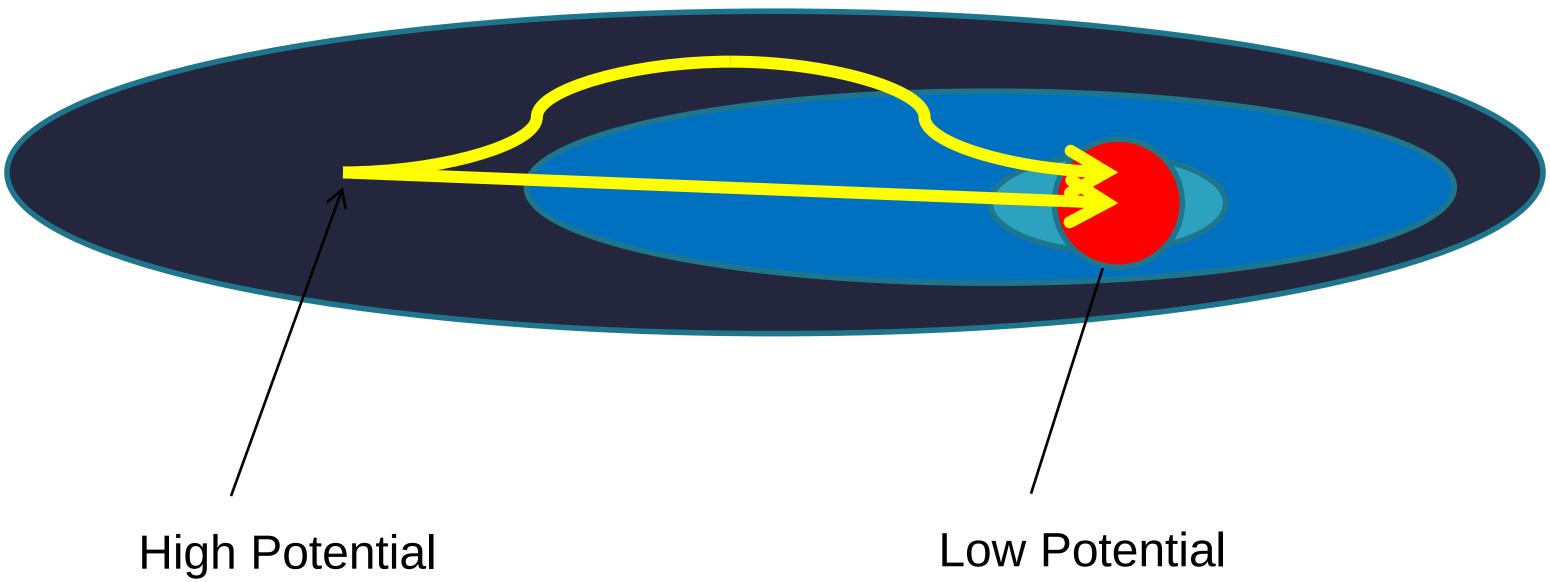
- The particle always travels in a way that
  - Respects inertia
  - Tries to minimize its potential



Energies and Forces

- Which trajectory to choose?
  - Initial velocity is the same as the “previous” velocity (that’s inertia)
  - Acceleration determined by the change in the potential, ie

$$m\vec{a} = \vec{F} = -\nabla V$$



Lagrangian Formulation

- That’s the energy formulation – now onto the Lagrangian formulation.

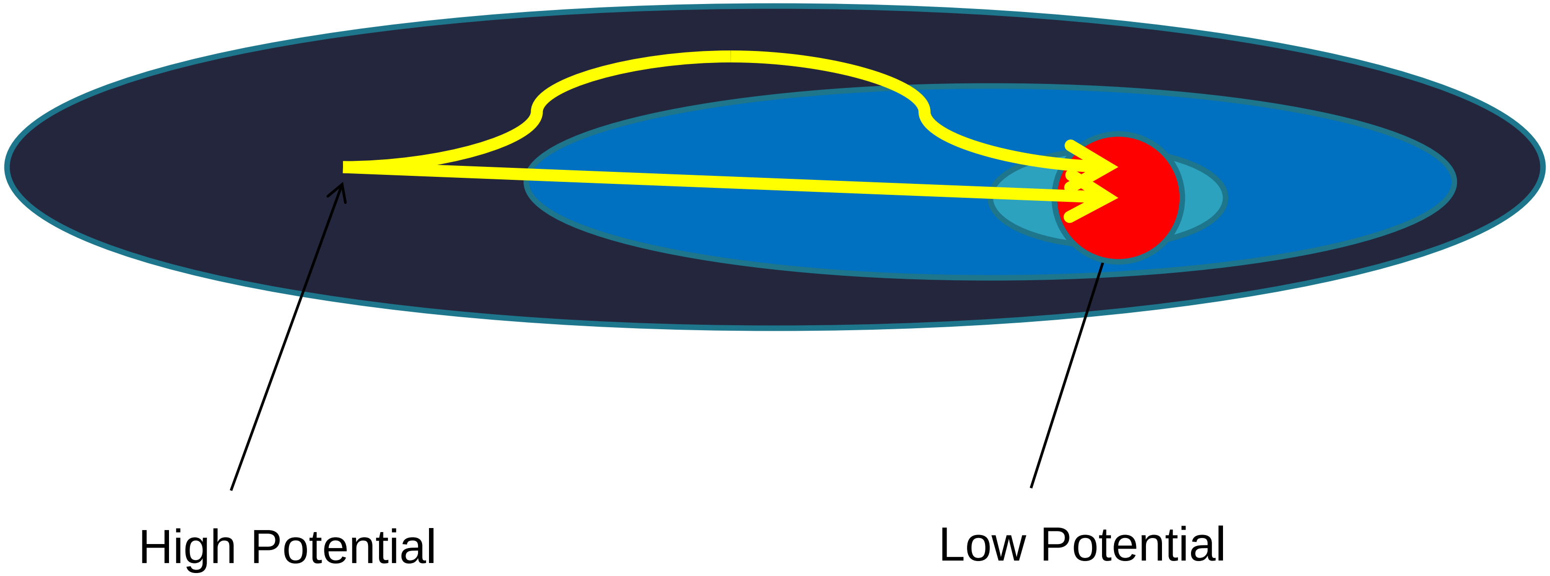
- This is a *formulation*. It gives no new information – there’s no *advantage* to it.

- But, easier than dealing with forces:
  - “generalized coordinates” – works with any convenient coordinates, don’t have to set up a coordinate system
  - Easy to read off symmetries and conservation laws

- The Lagrangian Formulation is based on *action*.

Action

- Here's the key idea: I want to spend as little time as possible in the region of high potential.
  - So, two competing goals: low time and low potential. When there is conflict, which one wins?
  - **Answer: the one that has the lowest action.**



Action

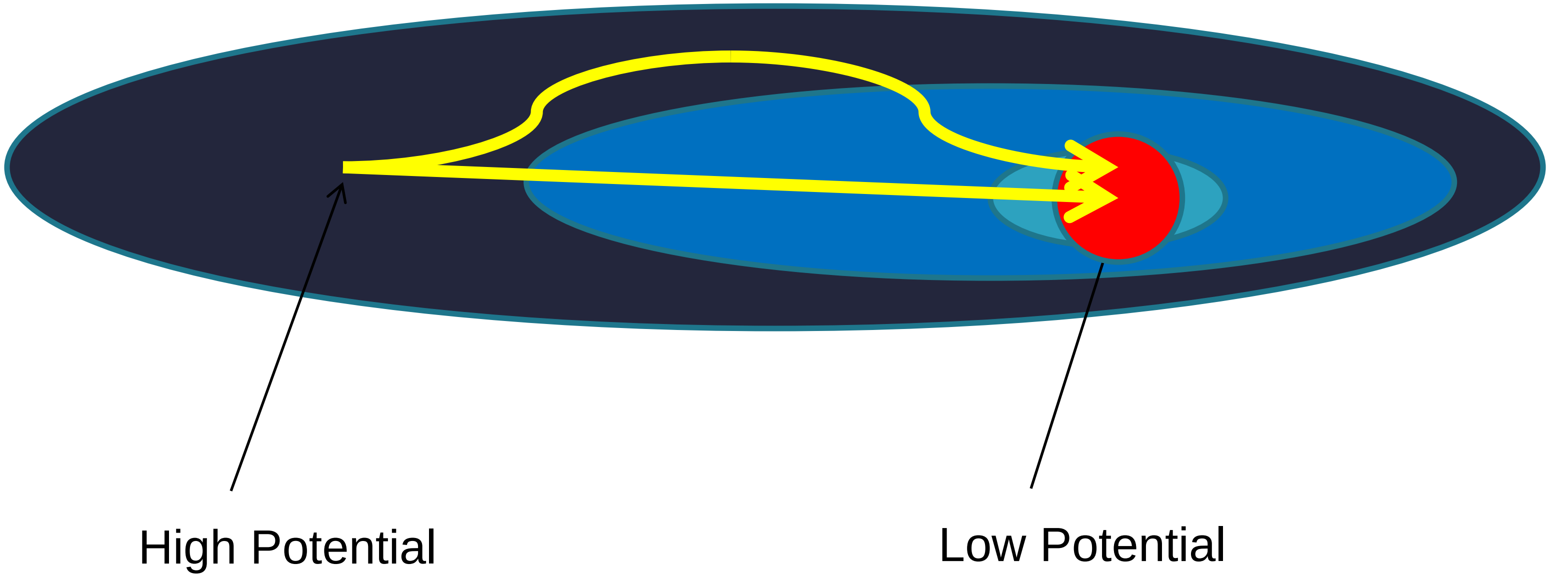
- Here's the idea:
  - You provide:
    - The fields as a function of time and space
    - All proposed trajectories (all infinity of them)
    - Any constraints
  - The action will tell you a number for each trajectory.
  - You choose the trajectory with the least action.

- So, action is a <u>functional</u>:

  Functions in → Numbers out

Action: an example

- Clearly, the lower trajectory has a smaller action, because it is the more direct path (“spend less time in regions of high potential”)



Action: Mathematical Definition

- S depends on the trajectory. The trajectory is completely defined by time, position, velocity, and the end points. Hence, S is written like this:

$$S\left[t, q_i(t), \dot{q}_i(t)\right]$$

Action

- So far, we can only compare two different possible trajectories. Is it possible to determine *a priori* the trajectory, without doing an infinite number of equations?
- Of course! **The proper trajectory will minimize the action.**
- Minimizing the action means that the derivative of the action will be zero on the proper trajectory.

Derivative of the Action

- The derivative of the action is called the Lagrangian, where the trajectory is parameterized by the time

$$S = \int L dt$$

- Sometimes, especially when dealing with the behavior of a field, it's more useful to take the derivative with respect to space as well. This is called the Lagrangian Density:

$$S = \int \mathscr{L} dt d^3x$$

The Lagrangian

- In classical mechanics, the Lagrangian has a simple definition:

$$L = T - V$$

- In field theory, the Lagrangian Density is defined similarly. For example, a free, classical electromagnetic field has

$$L = F^{uv}F_{uv}$$

The Equations of Motion

- The action depends on several variables (for example: x, the derivative of x, and t). Setting the derivative of the action equal to zero involves using the chain rule. The result is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

The Equations of Motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- Two very important notes:
  - There may be many q’s, in which case there are many copies of these equation, one for each q.
  - q and q dot are considered separate variables, parameterized by time. So:

$$\frac{\partial}{\partial q}\dot{q} = 0 \qquad \frac{\partial}{\partial \dot{q}}\dot{q} = 1 \qquad \frac{d}{dt}\dot{q} = \ddot{q}$$

Equations of Motion: Example

- I don’t want to derive the equations of motion, but here’s an example:

I’m dropped out of an airplane. The Lagrangian is:

$$L = T - V = \frac{1}{2}mv^2 - mgz$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Equations of Motion: Example

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

- Recall that the equation of motion is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- In the x-direction,

$$\ddot{x} = 0$$

Equations of Motion: Example

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

- Recall that the equation of motion is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- In the y-direction,

$$\ddot{y} = 0$$

Equations of Motion: Example

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

- Recall that the equation of motion is:

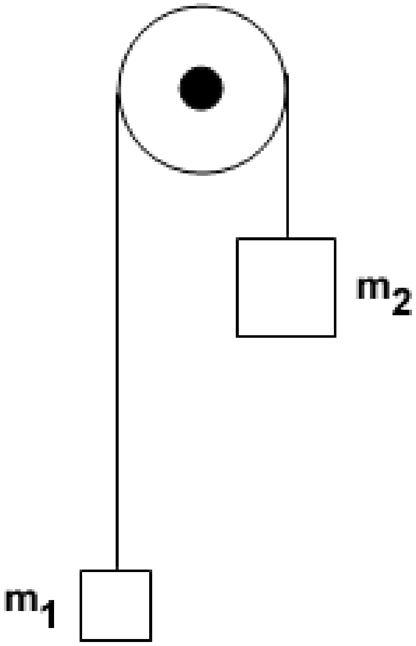
$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- In the z-direction,

$$\ddot{z} = -g$$

- So we get the answer that we would expect.

Equations of Motion: Nontrivial Example

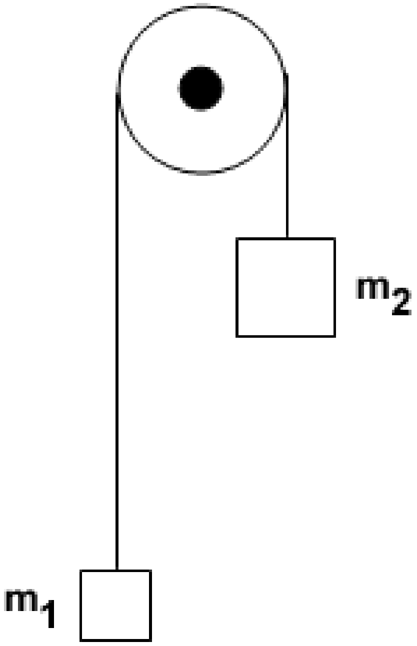


- Take M_1 to have position x, and the rope to have length L.
- $V = -m_1 g x - m_2 g(l-x)$
- $T = .5*m_1*v_x^2 + .5*m_2*v_x^2$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(l - x)$$

Equations of Motion: Nontrivial Example

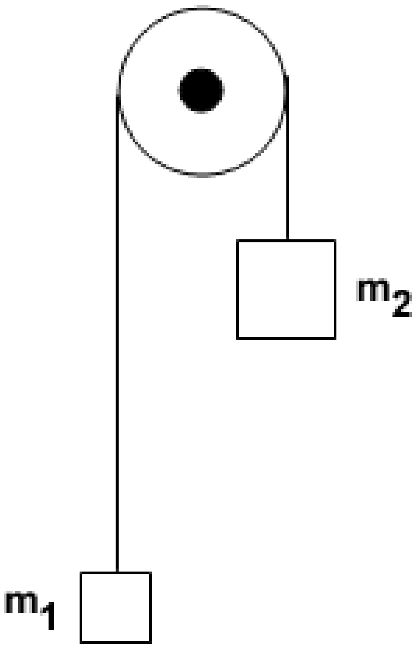
$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(l - x)$$



$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

Equations of Motion: Nontrivial Example



$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

The equation of motion is therefore:

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

Fun facts about the Lagrangian and the Action

- Conjugate variables:
  - Def’n: two variables that are Fourier Transforms of each other (like position and momentum in quantum mechanics).
  - If you take the derivative of the action with respect to q, the result is p, such that p and q are conjugate variables
  - Your generalized coordinates (the q’s) can be transformed into “canonically conjugate variables” by the Hamilton-Jacobi equations.
  - For now this is just a fun fact, but will be very useful if you study Hamiltonian Dynamics

Fun facts about the Lagrangian and the Action

- Symmetries:
  - Noether’s Theorem: if the Lagrangian doesn’t change under some transformation, than that transformation is a symmetry.
    - For example, my jumping out of a plane is symmetric in the x-direction, because the Lagrangian (below) doesn’t change if I take x → x + a

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

  - If the Lagrangian depends on the derivative of q, but not on q itself (as above), then q is called cyclic, and the “generalized momentum” (below) is conserved.

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

So, why are Lagrangians Good?

- Easy to find symmetries
- Get to use “generalized coordinates” – can be anything I want, even things that aren’t usually allowed (like angles)
- Easy to relate to more advanced methods, like Hamiltonians

- And best of all, they’re useful for fields!
  - How? Actually work in reverse – assume the equation of motion (Dirac Equation, Maxwell’s equation, etc), and cook up a Lagrangian that will yield the desired equation of motion
  - Then, use Lagrangian to get symmetries, path integrals, expectation values, projections, whatever you want

Example of a QFT Lagrangian

- Imagine a world with only one particle, a scalar named ϕ.

$$\mathscr{L} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{2} Z_g g \phi^3 + Y \phi$$

Each term represents a vertex:

- First (trivial) vertex: “particle can move”
- Second (trivial) vertex: “particle exists and has mass”
- Third (interaction) vertex: particle comes in and two particles go out.
- Fourth (trivial) vertex: particle lives by itself.

All of these vertices have vertex factors, chosen to obtain the desired behavior.

Lagrangians in Particle Physics

- You see the beauty here -- if we get the Lagrangian right, then all of physics follows. We can also look at little pieces of the Lagrangian to see more useful results

- Our "current" model for the universe is called the standard model. It is completely defined by a Lagrangian, which defines every particle in the universe and how they interact.

The Standard Model Lagrangian

$$
\begin{gathered}
-\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \frac{1}{2}ig_s^2(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \\
\bar{G}^a\partial^2G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu\partial_\nu W^-_\mu - M^2W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2Z^0_\mu Z^0_\mu - \\
\frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu - \frac{1}{2}\partial_\mu H\partial_\mu H - \frac{1}{2}m_h^2H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\frac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \frac{1}{2}g^2W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \\
\frac{1}{2}g^2W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2s_wc_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + \\
H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + \\
2(\phi^0)^2H^2] - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \\
\phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \\
\phi^0\partial_\mu H) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \\
\phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \\
\frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2-1)^2\phi^+\phi^-] - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \\
\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- - g^1s_w^2A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \\
\bar\nu^\lambda\gamma\partial\nu^\lambda - \bar u^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar d^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_wA_\mu[-(\bar e^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar u^\lambda_j\gamma^\mu u^\lambda_j) - \\
\frac{1}{3}(\bar d^\lambda_j\gamma^\mu d^\lambda_j)] + \frac{ig}{4c_w}Z^0_\mu[(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar e^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda) + (\bar u^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - \\
1-\gamma^5)u^\lambda_j) + (\bar d^\lambda_j\gamma^\mu(1-\frac{8}{3}s_w^2-\gamma^5)d^\lambda_j)] + \frac{ig}{2\sqrt{2}}W^+_\mu[(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar u^\lambda_j\gamma^\mu(1+ \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \frac{ig}{2\sqrt{2}}W^-_\mu[(\bar e^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar d^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1+\gamma^5)u^\lambda_j)] + \frac{ig}{2\sqrt{2}}\frac{m_e^\lambda}{M}[-\phi^+(\bar\nu^\lambda(1- \\
\gamma^5)e^\lambda) + \phi^-(\bar e^\lambda(1+\gamma^5)\nu^\lambda)] - \frac{g}{2}\frac{m_e^\lambda}{M}[H(\bar e^\lambda e^\lambda) + i\phi^0(\bar e^\lambda\gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar u^\lambda_j C_{\lambda\kappa}(1- \\
\gamma^5)d^\kappa_j) + m_u^\lambda(\bar u^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \frac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1- \\
\gamma^5)u^\kappa_j] - \frac{g}{2}\frac{m_u^\lambda}{M}H(\bar u^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar d^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar u^\lambda_j\gamma^5u^\lambda_j) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar d^\lambda_j\gamma^5 d^\lambda_j) + \\
\bar X^+(\partial^2-M^2)X^+ + \bar X^-(\partial^2-M^2)X^- + \bar X^0(\partial^2-\frac{M^2}{c_w^2})X^0 + \bar Y\partial^2Y + igc_wW^+_\mu(\partial_\mu\bar X^0X^- - \\
\partial_\mu\bar X^+X^0) + igs_wW^+_\mu(\partial_\mu\bar YX^- - \partial_\mu\bar X^+Y) + igc_wW^-_\mu(\partial_\mu\bar X^-X^0 - \partial_\mu\bar X^0X^+) + \\
igs_wW^-_\mu(\partial_\mu\bar X^-Y - \partial_\mu\bar YX^+) + igc_wZ^0_\mu(\partial_\mu\bar X^+X^+ - \partial_\mu\bar X^-X^-) + igs_wA_\mu(\partial_\mu\bar X^+X^+ - \\
\partial_\mu\bar X^-X^-) - \frac{1}{2}gM[\bar X^+X^+H + \bar X^-X^-H + \frac{1}{c_w^2}\bar X^0X^0H] + \frac{1-2c_w^2}{2c_w}igM[\bar X^+X^0\phi^+ - \\
\bar X^-X^0\phi^-] + \frac{1}{2c_w}igM[\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-] + igMs_w[\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-] + \\
\frac{1}{2}igM[\bar X^+X^+\phi^0 - \bar X^-X^-\phi^0]
\end{gathered}
$$

Conclusions

- There’s a famous sentiment: that the universe is simple and elegant. Based on the previous Lagrangian, that’s bull!

- Actually, it’s worse than it looks! Many of the constants given as “c” are actually irrational and baseless numbers

- And it’s wrong! Doesn’t account for matter dominance, lack of right-handed neutrinos, or neutrino mass – not to mention gravity!

- But, it actually works darn well, considering that all of physics is written in this one equation!