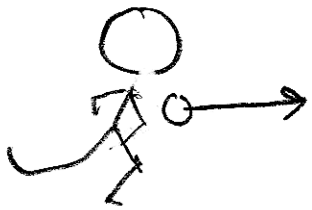
Addition of Velocities (∥ to relative velocity)

Runners Rest Frame

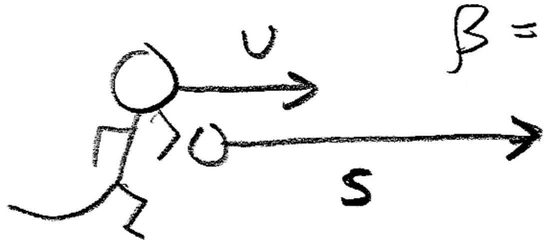


ball, speed w

$$\beta_b = \frac{w}{c}$$

$$x_b' = wt' = \beta_b ct'$$

Our rest frame



$$\beta = \frac{u}{c}$$

$$\beta_s = \frac{s}{c}$$

non relativistic case: $s = u + w$

Relativistically:
- Length contraction. (ball has less velocity)
- time dilitation (even less!)

$$x_b = x_b' + \beta ct' = \frac{(\beta_b + \beta)ct'}{\sqrt{1-\beta^2}}$$

$$t = \frac{t' + \beta \frac{x_b'}{c}}{\sqrt{1-\beta^2}} = \frac{(1 + \beta\beta_b)t'}{\sqrt{1-\beta^2}}$$

$$s = \frac{x_b}{t} = \frac{\beta_b + \beta}{1 + \beta\beta_b} \cdot c, \quad \boxed{\beta_s = \frac{s}{c} = \frac{\beta_b + \beta}{1 + \beta\beta_b}}$$

$$\beta_b = 1 - \varepsilon_b \qquad \beta = 1 - \varepsilon$$

$$\beta_s = \frac{1-\varepsilon_b + 1 - \varepsilon}{1 + (1-\varepsilon)(1-\varepsilon_b)} = \frac{2-(\varepsilon+\varepsilon_b)}{2-(\varepsilon+\varepsilon_b)+\varepsilon\varepsilon_b}$$

$$= \frac{1}{1 + \frac{\varepsilon\varepsilon_b}{2-(\varepsilon+\varepsilon_b)}} < 1$$

imagine $\varepsilon \ll 1$, $\varepsilon_b \ll 1$

$$\beta_s \cong \frac{1}{1+\frac{\varepsilon\varepsilon_b}{2}} \simeq 1 - \frac{\varepsilon\varepsilon_b}{2}$$

suppose

$$\beta_b = 0.99 \qquad \beta = 0.99$$

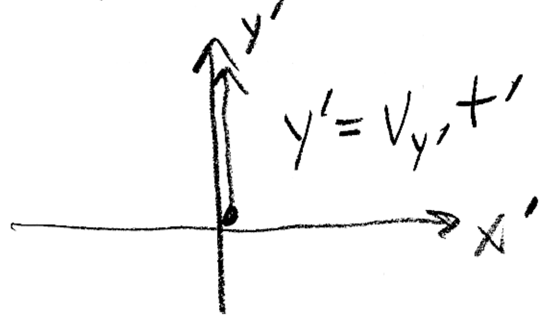
$$\beta_s \simeq 1 - \frac{1}{2}(0.01)(0.01)$$

$$\simeq 1 - \frac{1}{2}\cdot 10^{-4} = 1 - 5\cdot 10^{-5}$$

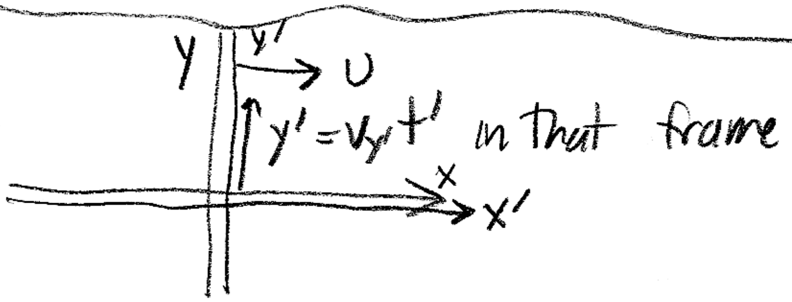
$$\boxed{\beta_s \simeq 0.99995}$$

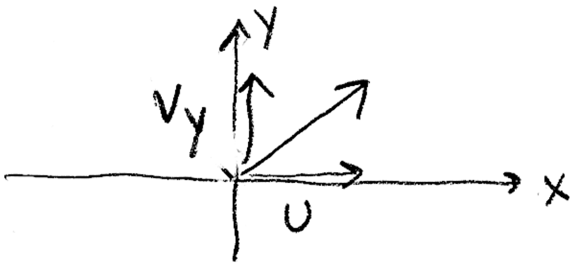
$\perp$ velocity transformation

"primed frame"



$x' = 0$





$$y = y' = v_{y'} t' = v_y t$$

$$t = \frac{t' + \frac{u}{c}\frac{x'}{c}}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = \frac{t'}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

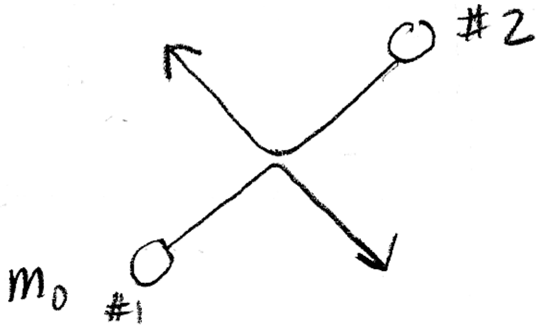
$$t' = \sqrt{1-\left(\frac{u}{c}\right)^2}\, t$$

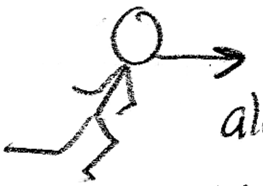
$$v_{y'} \cdot \sqrt{1-\left(\frac{u}{c}\right)^2}\, t = v_y t$$

$$\boxed{v_y = v_{y'} \cdot \sqrt{1-\left(\frac{u}{c}\right)^2}}$$

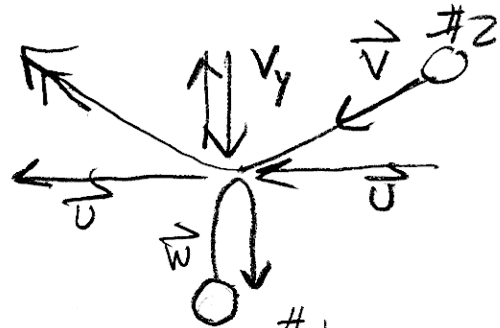
$\perp$ velocity contraction.

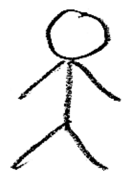
Relativistic Mass, Momentum : "rest mass" $\equiv m_0$





run alongside watch, so #1 has no velocity in horizontal





idea : $m(w)$

$m(v)$

$$v_y = w \cdot \sqrt{1-\left(\frac{u}{c}\right)^2}$$

$$\neq w$$

$\Rightarrow$ vertical velocity components are unequal, even though masses are same!

$\Rightarrow$ momenta must be equal though!

$$m(v)\, v_y = m(w)\, w$$

$$m(v)\, \cancel{w} \sqrt{1-\left(\frac{u}{c}\right)^2} = m(w)\, \cancel{w}$$

$$\frac{m(v)}{m(w)} = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

$$\lim_{w\to 0} m(w) = m_0$$

$$\lim_{w\to 0} v = u$$

$$\lim_{v \to 0} m(v) = m(0)$$

$$\boxed{m(v) = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$$

$$m = \frac{m_0}{\sqrt{1-\beta^2}} = \gamma m_0$$

<u>note</u>:

$$\lim_{v \to 0} m(v) = \frac{m_0}{1-\frac{1}{2}\left(\frac{v}{c}\right)^2} \simeq m_0\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2\right)$$

$$m(v)\cdot c^2 = m_0 c^2 + \underbrace{\frac{1}{2} m_0 v^2}_{\text{kinetic energy}}$$

interpret this as energy that remains when $v \to 0$

interpret this as total energy.

$$\boxed{E = mc^2 = m_0 \gamma c^2}$$

4 vectors

$\left(t, \frac{x}{c}, \frac{y}{c}, \frac{z}{c}\right) \leftarrow$ "4 vector"

$$t^2 - \left(\frac{x}{c}\right)^2 - \left(\frac{y}{c}\right)^2 - \left(\frac{z}{c}\right)^2 = \text{invariant}$$

$$= t'^2 - \left(\frac{x'}{c}\right)^2 - \left(\frac{y'}{c}\right)^2 - \left(\frac{z'}{c}\right)^2$$

Claim

$\left(\frac{E}{c^2}, \frac{p_x}{c}, \frac{p_y}{c}, \frac{p_z}{c}\right)$ is also a 4-vector.

$\frac{E}{c^2} = \frac{m_0 \gamma c^2}{c^2}$ $\quad p_x = m_0 \gamma v$ $\quad p_y = p_z = 0$ by definition. (choose x-direction)

then, $\left(\frac{E}{c^2}\right)^2 - \left(\frac{p_x}{c}\right)^2$ $\quad \beta = \frac{v}{c}$

$$= (m_0\gamma)^2 - (m_0\gamma\beta)^2 = m_0^2\gamma^2(1-\beta^2)$$

$$\left(\frac{E}{c^2}\right)^2 - \left(\frac{p_x}{c}\right)^2 = m_0^2 \cdot \frac{1-\beta^2}{1-\beta^2} = m_0^2$$

$$\left(\frac{E}{c^2}\right)^2 - \left(\frac{p_x}{c}\right)^2 = \left(\frac{E'}{c^2}\right) - \left(\frac{p_x'}{c}\right)^2 = m_0^2$$

always > 0

means:

$$\boxed{\begin{aligned} p_x' &= \gamma\left(p_x - \beta\frac{E'}{c}\right) \\ p_y' &= p_y \\ p_z' &= p_z \\ E' &= \gamma(E - \beta c p_x) \end{aligned}}$$

like x, y, z (for p_x', p_y', p_z')

like t (for E')