The Pole Vaulter Paradox and Proper Time

Note from Lorentz Transformation...

$$x' = \frac{x - ut}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

$$\left.\begin{array}{l} y' = y \\ z' = z \end{array}\right\}$$ neglect for moment, uninteresting

$$t' = \frac{t - \left(\frac{u}{c}\right)\left(\frac{x}{c}\right)}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

<u>Nicer units</u>: define $\beta \equiv \frac{u}{c}$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

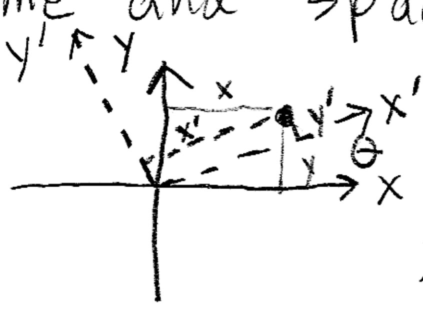
$$\left.\begin{array}{l} x' = \gamma(x - \beta ct) \\ t' = \gamma\left(t - \beta\left(\frac{x}{c}\right)\right) \end{array}\right\}$$

$c = 0.3 \frac{m}{\text{nanoseconds}}$

$= 0.3 \frac{m}{ns}$

work relativity in nanoseconds.

conceptually, resemble "rotation" of time and space.



$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$x'^2 + y'^2 = x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta$$
$$+ x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta$$

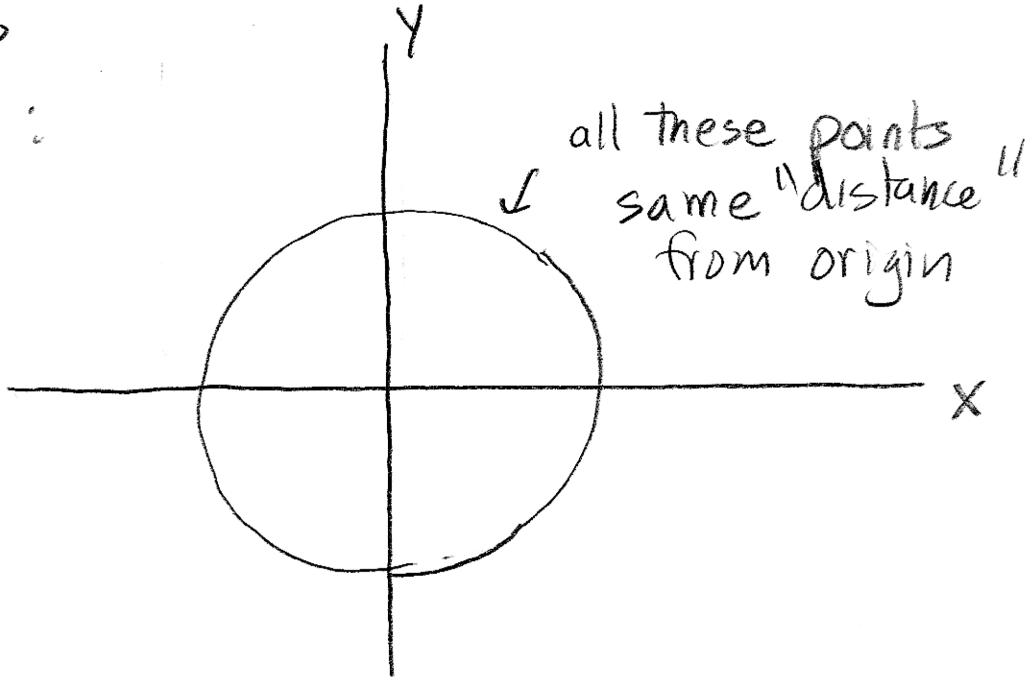
$$r^2 = x'^2 + y'^2 = x^2 + y^2$$ = independent of angle of rotation of coord system

<u>Moreso</u>:

$$d_{12}^2 = \underbrace{(x_1 - x_2)^2 + (y_1 - y_2)^2}_{\text{in one system}} = \underbrace{(x_1' - x_2')^2 + (y_1' - y_2')^2}_{\text{same in rotated coordinate system}}$$

$\uparrow$ distance between 2 vectors

<u>Visually</u>:



Things change for the Lorentz Transformation

$$t'^2 - \left(\frac{x'}{c}\right)^2 = \gamma^2\left(t - \beta\left(\frac{x}{c}\right)\right)^2 - \gamma^2\left(\frac{x}{c} - \beta t\right)^2$$

note the minus!

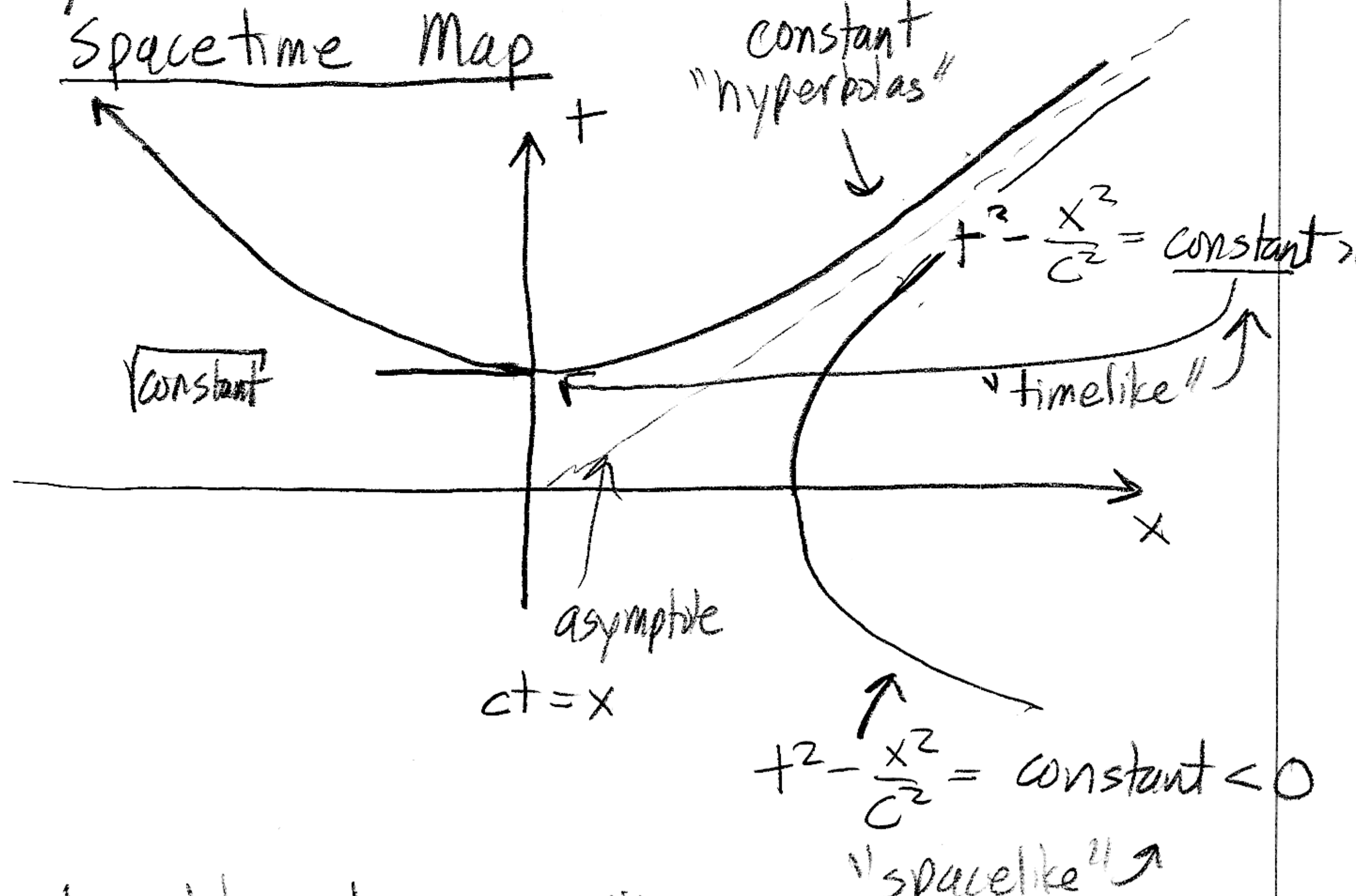
$$= \gamma^2\left[t^2 - \cancel{2\beta\left(\frac{x}{c}\right)t} + \beta^2\left(\frac{x}{c}\right)^2 - \left(\frac{x}{c}\right)^2 + \cancel{2\beta\left(\frac{x}{c}\right)t} - \beta^2 t^2\right]$$

$$= \frac{1}{1-\beta^2}\left[(1-\beta^2)t^2 - (1-\beta^2)\left(\frac{x}{c}\right)^2\right]$$

$$\boxed{t'^2 - \left(\frac{x'}{c}\right)^2 = \underbrace{t^2 - \left(\frac{x}{c}\right)^2}_{\text{"The invariant"}}}$$

- sign super important!

Spacetime Map



timelike + spacelike usually pertain to distance between pairs of events:

Event #1 : (t_1, x_1) #2 (t_2, x_2)

$$(t_1 - t_2)^2 - \frac{1}{c^2}(x_1 - x_2)^2 = \text{"the interval"}$$

when $\Delta t^2 - \frac{1}{c^2}\Delta x^2 > 0$ "timelike"

In some inertial frame, $\Delta x' = 0$

in that frame $(\Delta t')^2 = \text{"the interval"}$

$\Delta t' = \sqrt{\text{"the interval"}} \equiv$ the proper time.

AKA τ

Example:

Event #1: #2

$t_1 = 0$ $t_2 = 5$ ns

$x_1 = 0$ $x_2 = 0.9$ m

$$\text{Interval} = \tau^2 = (5-0)^2 - \left(\frac{0.9\text{ m}}{0.3\text{ m/ns}}\right)^2$$

$$= 25^2 - 3^2 \quad \text{ns}^2$$

$$\tau^2 = 16 = 4^2 \text{ ns}^2$$

$$\tau = 4\text{ ns}$$

There exists a frame where $x_1' = x_2'$ and time interval $t_2' - t_1' = 4$ ns. What frame is that? One moving with speed $\frac{x_2}{t_2} = \frac{0.9}{5} = 0.14$ m/ns, so things initially at $x_1' = 0$ stay at $x_2' = 0$, $\Delta x' = 0$

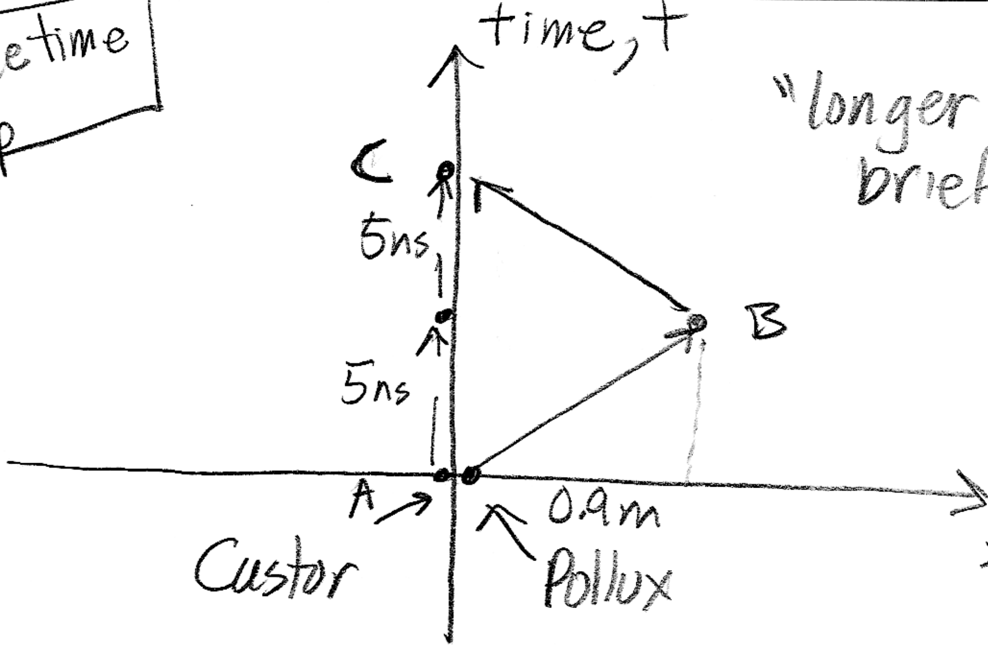
$\Rightarrow$ stare at $\Delta t^2 - \frac{\Delta x^2}{c^2} =$ constant

shortest when $\Delta x = 0$.

Time moves slowest when the clock is at rest ($\Delta x = 0$) w/r to you.

TWIN "PARADOX"

Spacetime Map



"longer is briefer"

Castor stays at $x=0$, goes from $A \to C$, That is, stays put, for 10 ns

Pollux goes $A \to B$, $B \to C$; lots of acceleration at A/B/C.

How much time goes by on a clock that rides with Pollux?

$\Rightarrow$ look at intervals:

$A \to B$ $\quad \Delta t^2 - \frac{\Delta x^2}{c^2} = 5^2 - \left(\frac{0.9}{0.3}\right)^2 \text{ ns}^2$

$(\text{Interval})^2 = 4^2 \text{ ns}$

$\tau = (\text{Interval}) = 4 \text{ ns}$

$B \to C$ also 4 ns

Pollux: $\boxed{A \to B \to C, \quad 8 \text{ ns}}$

Castor: $\boxed{A \to C, \quad 10 \text{ ns}}$