The Pole Vaulter Paradox and Proper Time

Note from Lorentz Transformation:

\[ x' = \frac{x - ut}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} \]

\[ y' = y \]

\[ z' = z \] (neglect for moment, uninteresting)

\[ t' = \frac{t - \frac{u}{c}z}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} \]

**Nicer units:** define \( \beta = \frac{u}{c} \)

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]

\[ x' = \gamma(x - \beta ct) \]

\[ t' = \gamma(t - \beta \frac{x}{c}) \]

constant \( c = 0.3 \ \text{m/s} \),

\[ t' = 0.3 \ \text{ns} \]

work relativinity in nanoseconds.

Conceptually, resemble "rotation" of time and space.

\[ x' = x \cos \theta + y \sin \theta \]

\[ y' = -x \sin \theta + y \cos \theta \]

\[ x'^2 + y'^2 = x^2 \cos^2 \theta + 2xy \sin \theta \cos \theta + y^2 \sin^2 \theta \]

\[ + x^2 \sin^2 \theta - 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta \]
\[ r^2 = x'^2 + y'^2 = x^2 + y^2 \text{ is independent of angle of rotation of system} \]

More so:
\[ d_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 \]

\( \uparrow \) distance between 2 vectors in one system same in rotated coordinate system

Visually:

![Diagram](image)

all these points same "distance" from origin

Things change for the Lorentz Transformation:

\[ t'^2 - \left(\frac{x'}{c}\right)^2 = \gamma^2 \left( t - \beta \frac{x}{c} \right)^2 = \gamma^2 \left( \frac{x}{c} - \beta t \right)^2 \]

Note the
\[ = \gamma^2 \left[ t^2 - 2 \beta \frac{x}{c} t + \beta^2 \frac{x^2}{c^2} - \left( \frac{x}{c} \right)^2 + 2 \beta (\frac{x}{c}) t - \beta^2 \frac{x^2}{c^2} \right] \]

minus!
\[ = \frac{1}{\gamma^2} \left[ (1 - \beta^2) t^2 - (1 - \beta^2) \left( \frac{x}{c} \right)^2 \right] \]

\[ t'^2 - \left(\frac{x'}{c}\right)^2 = t^2 - \left(\frac{x}{c}\right)^2 \]

"The invariant!"
In some inertial frame, \( \Delta t' = \sqrt{1 - \frac{1}{c^2} \Delta x^2} \) is "the interval" and \( \Delta x' = 0 \).

When \( \Delta t^2 - \frac{1}{c^2} \Delta x^2 > 0 \), time-like pairs of events:

Event #1: \((t_{1}, x_{1})\)

Event #2: \((t_{2}, x_{2})\)

Time-like: as distance between space-like: as constant.<br>Space-like: as constant.\[c^2 = x^2 + t^2\]
Example:

Event #1:  #2
\[ t_1 = 0 \quad t_2 = 5 \text{ ns} \]
\[ x_1 = 0 \quad x_2 = 0.9 \text{ m} \]

Interval:
\[ \tau^2 = (5-0)^2 - (\frac{0.9 \text{ m}}{0.3 \text{ m/ns}})^2 \]
\[ = 25^2 - 3^2 \quad \text{ns}^2 \]
\[ \tau^2 = 16 = 4^2 \quad \text{ns}^2 \]
\[ \tau = 4 \text{ ns} \]

There exists a frame where \( x_1' = x_2' \) and time interval \( t_2' - t_1' = 4 \text{ ns} \). What frame is that? One moving with speed \( \frac{x_2}{t_2} = \frac{0.9}{5} = 0.14 \text{ m/ns} \), so things initially at \( x_1' = 0 \) stay at \( x_2' = 0 \), \( \Delta x' = 0 \)

\[ \Rightarrow \text{ stare at } \Delta t^2 - \frac{\Delta x^2}{c^2} = \text{ constant} \]

shortest when \( \Delta x = 0 \).

Time moves slowest when the clock is at rest \( (\Delta x = 0) \) w/r to you.

TWIN “PARADOX”
Castor stays at $x=0$, goes from $A \rightarrow C$, that is, stays put for 10 ns.

Pollux goes $A \rightarrow B$, $B \rightarrow C$; lots of acceleration at $A/B/C$.

How much time goes by on a clock that rides with Pollux?

⇒ look at intervals:

$A \rightarrow B$: $\Delta t^2 = \frac{\Delta x^2}{c^2} = 5^2 - \left(\frac{0.9}{0.3}\right)^2 \text{ ns}^2$

$(\text{Interval})^2 = 4 \text{ ns}$

$\Delta t = (\text{Interval}) = 4 \text{ ns}$

$B \rightarrow C$ also 4 ns

Pollux: $A \rightarrow B \rightarrow C$, 8 ns

Castor: $A \rightarrow C$, 10 ns