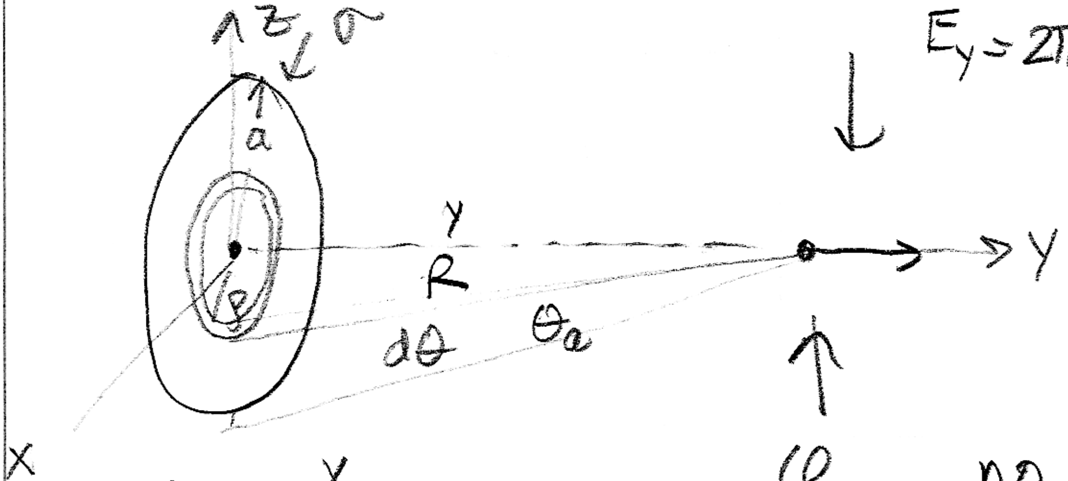
Disk:

earlier, $dE_y = 2\pi\sigma \sin\theta\, d\theta$

$E_y = 2\pi\sigma(1-\cos\theta_a)$



φ ... no angles

$$\cos\theta_a = \frac{y}{\sqrt{a^2+y^2}}$$

$$d\varphi = \frac{(2\pi\rho\, d\rho)\cdot\sigma}{\left(\frac{y}{\cos\theta}\right)} = \frac{dq}{R}$$

$$\rho = y\tan\theta \qquad d\rho = \frac{y}{\cos^2\theta}\, d\theta$$

$$d\varphi = \frac{2\pi\cdot y\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{y}{\cos^2\theta}\,\sigma\, d\theta}{\frac{y}{\cos\theta}}$$

$$d\varphi = 2\pi\sigma y\,\frac{\sin\theta}{\cos^2\theta}\, d\theta$$

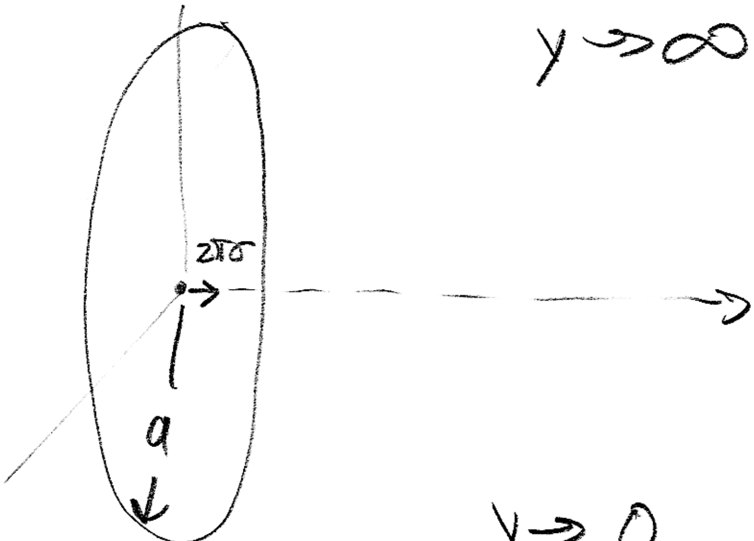
$$\varphi = 2\pi\sigma y\int_0^{\theta_a}\frac{\sin\theta}{\cos^2\theta}\, d\theta = 2\pi\sigma y\left[+\frac{1}{\cos\theta}\right]_0^{\theta_a}$$

$$= 2\pi\sigma y\left[\frac{1}{\cos\theta_a} - 1\right] \qquad \cos\theta_a = \frac{y}{\sqrt{a^2+y^2}}$$

Guesses!

$y \to 0, \quad E_y \to 2\pi\sigma$

$y \to \infty, \quad E_y = \dfrac{\pi \sigma a^2}{y^2}$



$y \to 0, \quad \varphi \to -2\pi\sigma y$

$y \to \infty, \quad \varphi \to \dfrac{\pi \sigma a^2}{y}$

$$E_y = 2\pi\sigma(1-\cos\theta_a)$$

$$\varphi = 2\pi\sigma y\left(-1 + \frac{1}{\cos\theta_a}\right)$$

∞ sheet:
$\theta_a \to \pi/2$
$\cos\theta_a \to 0$

on-axis

check

$$\varphi = 2\pi\sigma\left(\sqrt{y^2+a^2} - y\right)$$

$$\frac{\partial\varphi}{\partial y} = 2\pi\sigma\left(\frac{y}{\sqrt{y^2+a^2}} - 1\right) = 2\pi\sigma(\cos\theta_a - 1)$$

$$E_y = -2\pi\sigma(\cos\theta_a - 1) = 2\pi\sigma(1-\cos\theta_a) \checkmark$$

Limits:

$y \to 0$, $\quad \varphi \to 2\pi\sigma a \quad$ (a finite)

$\qquad\qquad \varphi \to -2\pi\sigma y \to 0 \quad (a \to \infty)$

Physical Interpretation:

a finite: $\varphi(\infty) = 0$

$q\varphi(0)$ is work to bring charge from ∞ to 0

a infinite: define $\varphi(0) = 0$!

(otherwise would be ∞)

$y \to \infty \quad \sqrt{y^2+a^2} = y\sqrt{1+\left(\frac{a}{y}\right)^2} \simeq y\cdot\left(1+\frac{1}{2}\left(\frac{a}{y}\right)^2\right)$

$$\varphi \to 2\pi\sigma\left(y + \frac{1}{2}\frac{a^2}{y} - y\right)$$

$$\varphi \to \frac{\pi a^2 \sigma}{y} \quad \text{(as guessed)}$$

$$E_y = -\frac{\partial E_y}{\partial y} = -\frac{\pi a^2 \sigma}{-y^2} = \frac{\pi a^2 \sigma}{y}$$