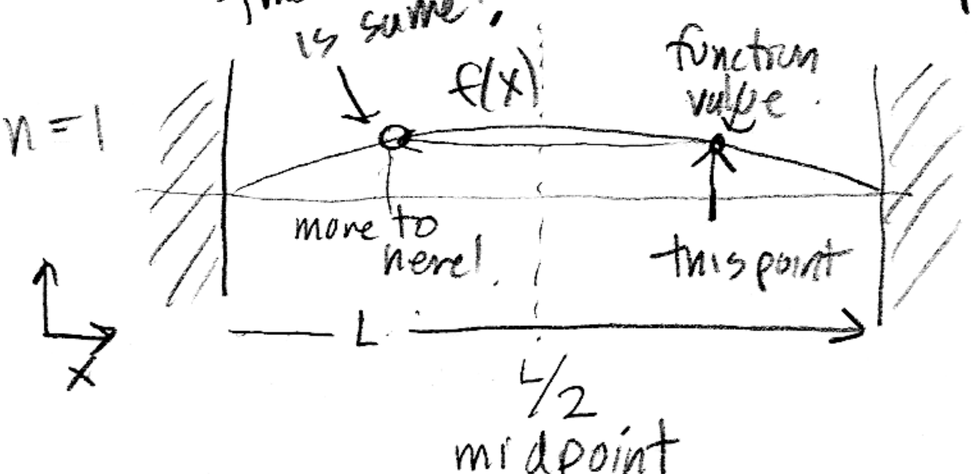
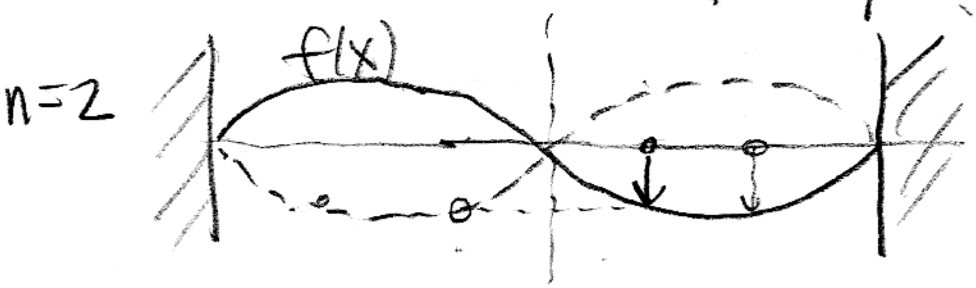
Two generic observations

(1) Parity ⇐ the operation of flipping about the midpoint



"Even" under parity flip

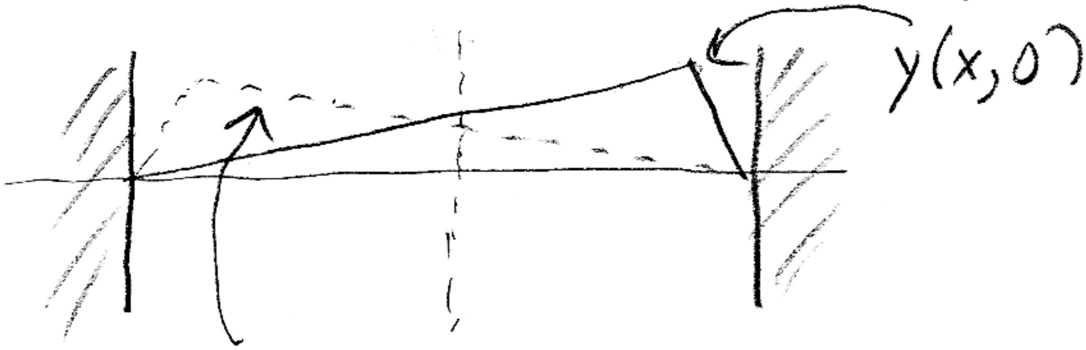


$$\underbrace{f_p(x)}_{\text{dotted line}} \neq f(x) = -f(x)$$

"odd" parity

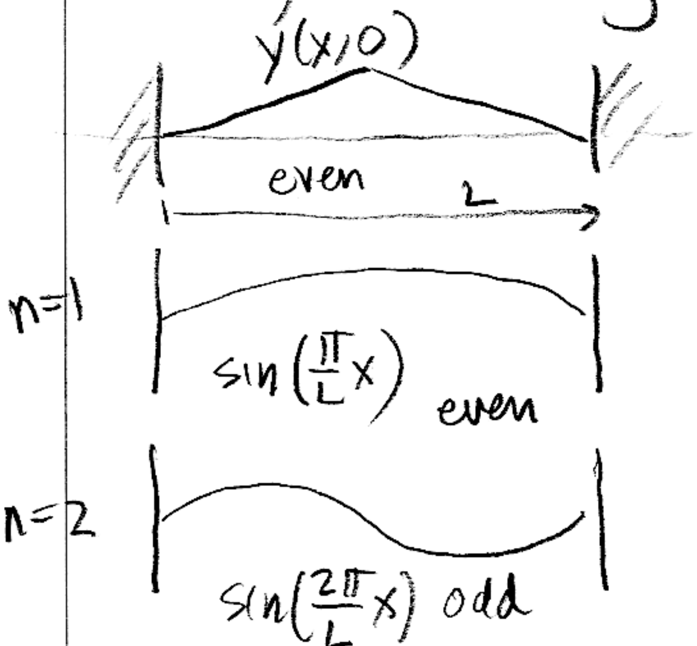
$$\boxed{P_n = \text{parity } n\text{th "harmonic"} = (-1)^{n-1}}$$

$y(x,0)$ may or may not have "definite" parity.



$$y_p(x,0) \neq y(x,0)$$
$$\neq -y(x,0)$$
$$\neq (\text{anything}) \times y(x,0)$$

When $y(x,0)$ is even or odd, something nice happens



$$B_n = \frac{2}{L}\int_0^L dx\, \underbrace{y(x,0)\sin\left(\frac{n\pi}{L}x\right)}$$

$$B_1 \neq 0 \quad (B_{(\text{odd}\#)} \neq 0)$$

$$B_2 = 0\ !$$

$$B_{(\text{even}\#)} = 0\ !$$

(Guitar)

② Time dependendence.

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$$

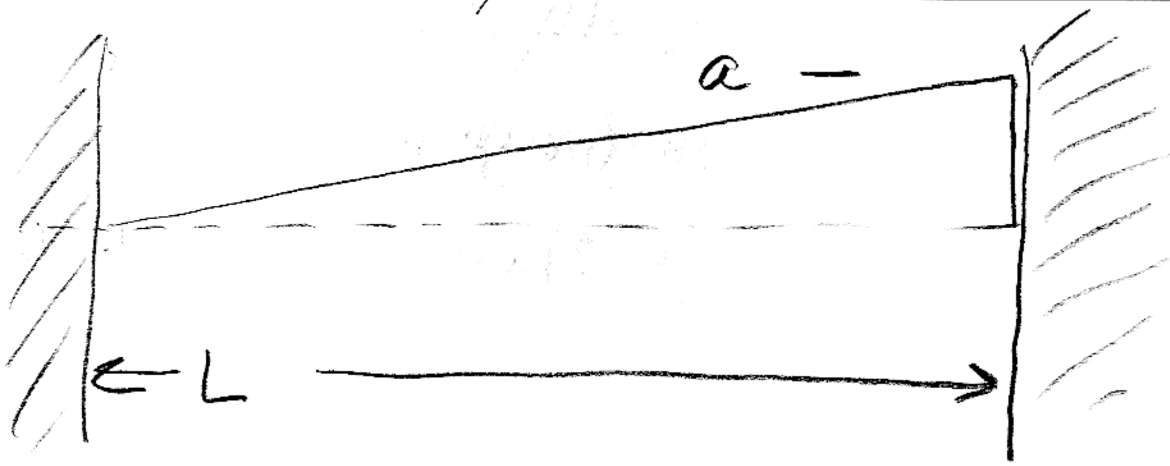
$$\omega_n = n\omega_1 = \frac{n\pi v}{L}$$

$$B_1 \sin\left(\frac{\pi}{L}x\right)\cos(\omega_1 t) + B_2 \sin\left(\frac{2\pi}{L}x\right)\underline{\cos(2\omega_1 t)}$$
$$+ B_3 \sin\left(\frac{3\pi}{L}x\right)\underline{\cos(3\omega_1 t)} + \ldots$$

not important for ear

mainly, you hear time dependence

"Height" on the fourier analysis does give you the "strength" of the harmonics.



$$y(x,0) = \frac{a}{L} \cdot x = \frac{a}{L} x$$

$$B_n = \frac{2}{L} \int_0^L dx \left(\frac{a}{L} x \right) \sin\left(\frac{n\pi}{L} x \right)$$

$$= \frac{2a}{L} \int_0^L dx \, x \sin\left(\frac{n\pi}{L} x \right)$$

$$z \equiv \frac{x}{L}$$

$$= 2aL \int_0^1 dz \, z \sin(n\pi z)$$

$u = z$ $\qquad dv = dz \sin(n\pi z)$

$du = dz$ $\qquad v = -\frac{1}{n\pi} \cos(n\pi z)$

$$\int_0^1 dz \, z \sin(n\pi z) = -\frac{z}{n\pi} \cos(n\pi z) \Big|_0^1 + \frac{1}{n\pi} \int_0^1 dz \cos(n\pi z)$$

$$= -\frac{1}{n\pi} \cos(n\pi) + \cancel{\frac{1}{(n\pi)^2} \sin(n\pi z) \Big|_0^1} \to 0$$

$\cos((0,2,4,\ldots)\pi) = 1$

$\cos((1,3,5,\ldots)\pi) = -1$

$$\boxed{B_n = \frac{2aL}{n\pi}(-1)^{n+1}}$$

So,

$$\boxed{\frac{a}{L}x = \frac{2aL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{L}x\right)}$$

Plot with Mathematica

Add time dependence.