Chapter 20 Wave Interence / Normal Modes

1. Superposition
2. Boundary Conditions
3. Standing Waves
4. Normal Modes
5. Interference
6. Resonance

(1–4: widely applicable, say, in quantum mechanics.)

---

1. Superposition: · wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

suppose $y_1(x,t)$ and $y_2(x,t)$ are two solutions to the wave equation:

ⓐ $\frac{\partial^2 y_1(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y_1(x,t)}{\partial x^2}$ ⓑ $\frac{\partial^2 y_2(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y_2(x,t)}{\partial x^2}$

then: $\alpha y_1(x,t) + \beta y_2(x,t)$ is also a solution of the wave equation.

#'s that are independent of x + t

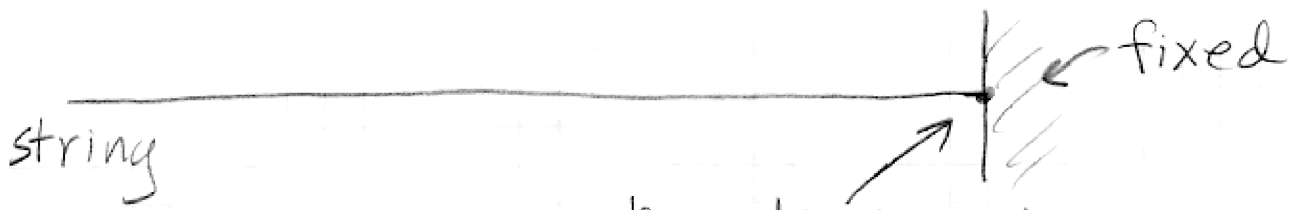
To see this, multiply ⓐ by α, ⓑ + β, add up, factor out.

$$\alpha \frac{\partial^2 y_1}{\partial t^2} = \alpha v^2 \frac{\partial^2 y_1}{\partial x^2}$$

$$\beta \frac{\partial^2 y_2}{\partial t^2} = \beta v^2 \frac{\partial^2 y_1}{\partial x^2}$$

$$\frac{\partial^2}{\partial t^2}(\alpha y_1 + \beta y_2) = v^2 \frac{\partial^2}{\partial x^2}(\alpha y_1 + \beta y_2)$$

What use is this? To visualize boundary conditions.

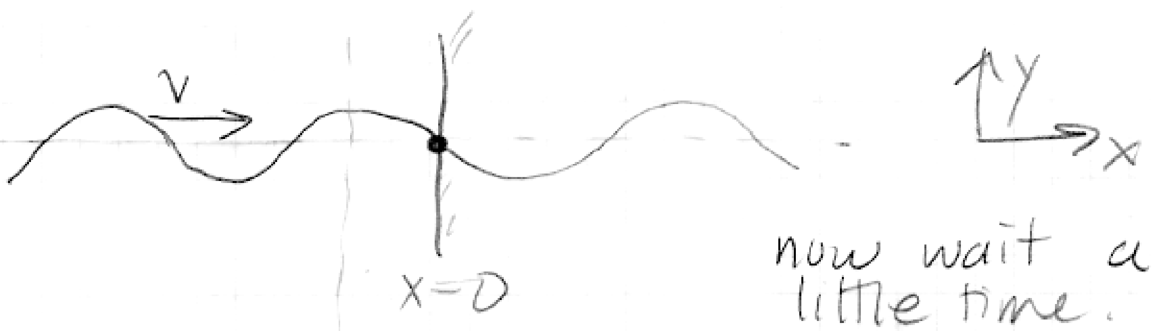


how do we describe the effect of this... <u>boundary</u> <u>condition</u>

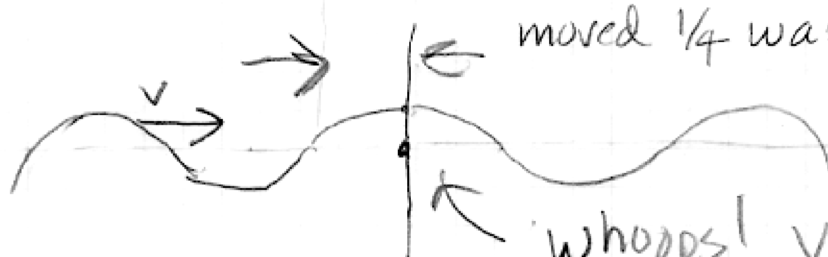
Try something... would $y(x,t) = A\sin(\omega t - kx)$ work? $\omega = \frac{2\pi}{T}$ (period), $k = \frac{2\pi}{\lambda}$ (wavelength)

(this is a solution to wave equation).

Put $x=0$... on the wall... $t=0$



now wait a little time.



moved 1/4 wavelength. $t = \frac{1}{4}T$

whoops! $y(0,t) \neq 0$

How to fix this: view the first guess as a $y_1(x,t)$ $(= A\sin(\omega t - kx))$, and add in a second solution $y_2(x,t)$, which maintains the boundary condition of $y(0,t) = 0$

$$y_1(0,t) + y_2(0,t) = 0$$

$$A\sin(\omega t - k\cdot 0) + y_2(0,t) = 0$$

$$y_2(0,t) = -A\sin(\omega t)$$

what about spatial dependence?

Given ωt, there are only two options that satisfy the wave equation:

$$y_2(x,t) = -A\sin(\omega t \pm kx)$$

so the total function will be...

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= A\left(\sin(\omega t - kx) - \sin(\omega t \pm kx)\right)$$

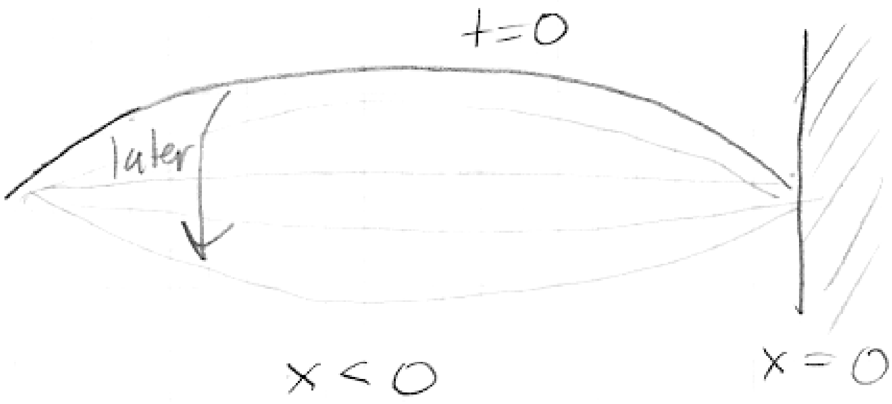
$= 0$ (trivial, uninteresting) ← − case

+ case

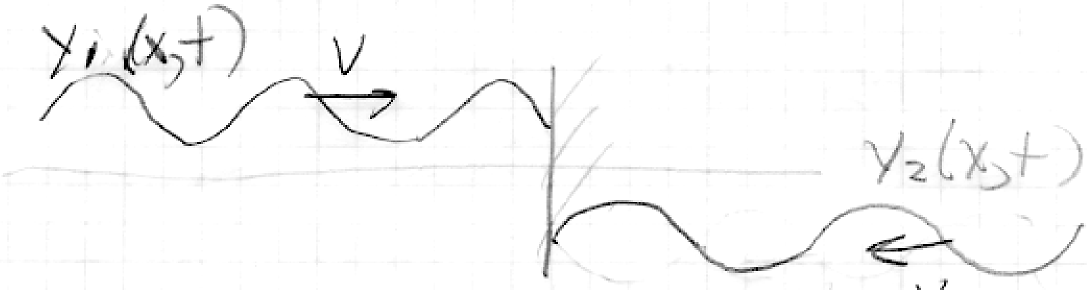
$$y(x,t) = A\left(\sin(\omega t - kx) - \sin(\omega t + kx)\right)$$

$$= A\left(\cancel{\sin(\omega t)\cos(-kx)} + \cos(\omega t)\sin(-kx) - \cancel{\sin(\omega t)\cos(kx)} - \cos(\omega t)\sin(kx)\right)$$

$$\boxed{y(x,t) = -2A\sin kx\cos\omega t}$$



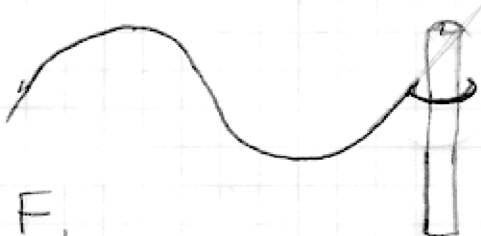
Visualization



upside down, moving in opposite direction

Add them together.

Another boundary condition "open end"



→ drawn with $\left.\frac{\partial y}{\partial x}\right|_{\text{boundary}} \neq 0$

tension F,
y component F_y
x component F_x

Frictionless Post, Massless Ring.

if so, $F_y = -F\left.\frac{\partial y}{\partial x}\right|_{\text{bound.}} \neq 0$

In this case, the ring (massless) would feel infinite acceleration.

⇒ non-physical.

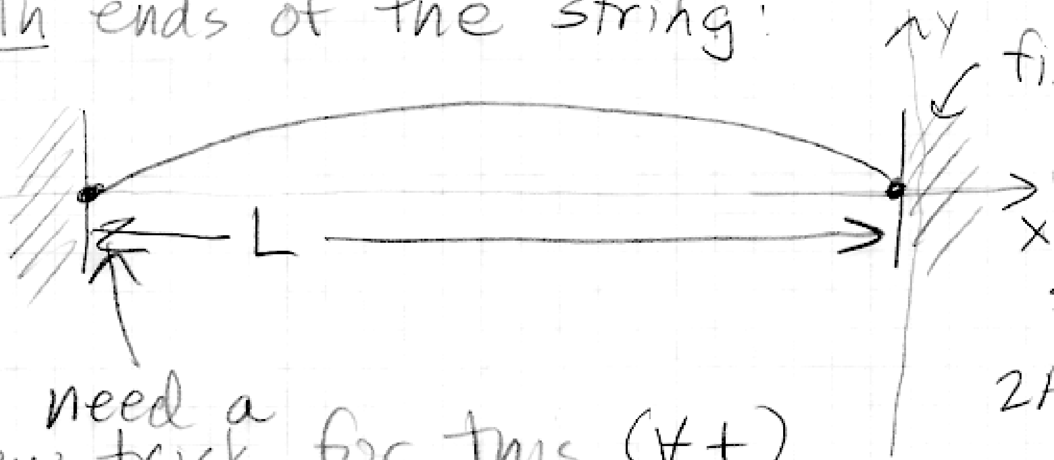
⇒ conclude: $\left.\frac{\partial y}{\partial x}\right|_{\text{boundary}} = 0$

$\underline{y(x,t)} = A(\sin(\omega t - kx) + \sin(\omega t + kx))$

$$\boxed{y(x,t) = 2A\cos(kx)\sin(\omega t)}$$

3. Standing Waves

Imagine now boundary conditions on <u>both</u> ends of the string:



(satisfied by superposition $2A\sin(kx)\cos(\omega t)$)

need a new trick for this ($\forall t$)

$$\sin(kL) = 0$$

$$kL = \underbrace{0}_{\text{trivial}},\ \underbrace{\pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \ldots}_{\text{- sign equivalent to + sign + time origin change}}$$

$$\frac{2\pi}{\lambda}L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, \ldots}$$

$$\lambda_n \cdot f_n = v$$

f_n = frequency (in <u>time</u> that) corresponding to λ_n

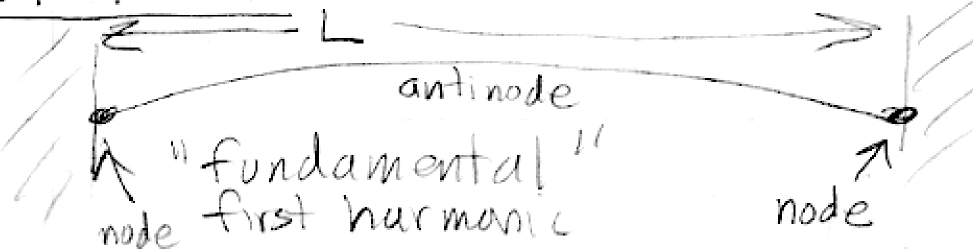
$$\boxed{f_n = \frac{nv}{2L}}$$

String: $v = \sqrt{\dfrac{F}{\mu}}$

Sketches

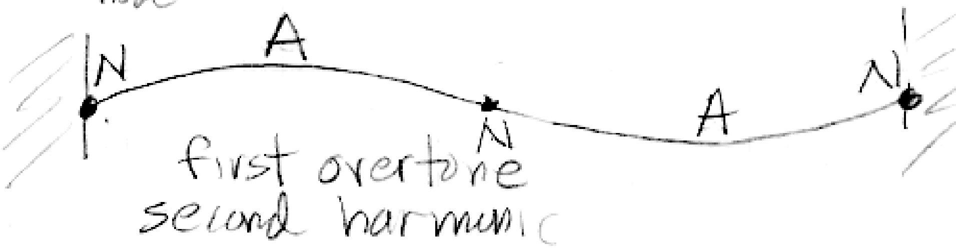
$n = 1 \quad f_1 = \dfrac{v}{2L}$

$\lambda_1 = 2L$



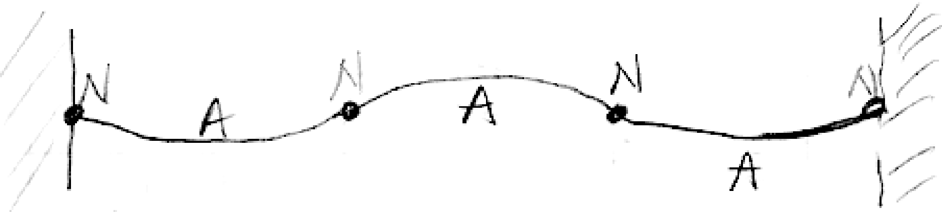
$n = 2 \quad f_2 = \dfrac{v}{L}$

$\lambda_2 = L$



$n = 3 \quad f_3 = \dfrac{2v}{3L}$

$\lambda_3 = \dfrac{3v}{2L}$



note: $n = \#$ antinodes

This sequence of functions...

$$\xi_1(x,t) = \alpha_1 \sin\left(\frac{2\pi}{\lambda_1}x\right)\cos(2\pi f_1 t)$$

normalization constant, more later

$$\xi_1(x,t) = \alpha_1 \sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi v}{L}t\right)$$

$$\vdots$$

$$\xi_n(x,t) = \alpha_n \sin\left(\frac{\pi n}{L}x\right)\cos\left(\frac{\pi n v}{L}t\right)$$

are called the "normal modes" of the string. They are infinitely many of them. They are like "unit vectors"

Normalization

biggest $\cos^2\left(\frac{\pi n v t}{2}\right)$ gets like

want: $$\int_{-L}^{0} dx\, \alpha_n^2 \sin^2\left(\frac{\pi n}{L}x\right) \times \underset{\uparrow}{1} = \underline{1} \begin{cases} \hat{i}^2 = 1 \\ \hat{j}^2 = 1 \end{cases}$$

$u = \sin\left(\frac{\pi n}{L}x\right)$ $\quad dv = dx \sin\left(\frac{\pi}{L}n x\right)$

→ parts integration. $\quad du = \left(\frac{\pi n}{L}\right)\cos\left(\frac{\pi n}{L}x\right)$ $\quad v = -\frac{L}{\pi n}\cos\left(\frac{\pi n}{L}x\right)$

$$\int_{-L}^{0} dx \sin^2\left(\frac{\pi n}{L}x\right) = \underbrace{-\frac{L}{\pi n}\sin\left(\frac{\pi n}{L}x\right)\cos\left(\frac{\pi n}{L}x\right)\Big|_{-L}^{0}}_{0} + \int_{-L}^{0} dx \underbrace{\cos^2\left(\frac{\pi n}{L}x\right)}_{1-\sin^2\left(\frac{\pi n}{L}x\right)}$$

$$2\int_{-L}^{0} dx \sin^2\left(\frac{\pi n}{L}x\right) = \int_{-L}^{0} dx = L$$

$$\int_{-L}^{0} dx \sin^2\left(\frac{\pi n}{L}x\right) = \frac{L}{2}$$

so $$\alpha_n^2 \int_{-L}^{0} dx \sin^2\left(\frac{\pi n}{L}x\right) = 1$$

$$\alpha_n^2 \times \frac{L}{2} = 1$$

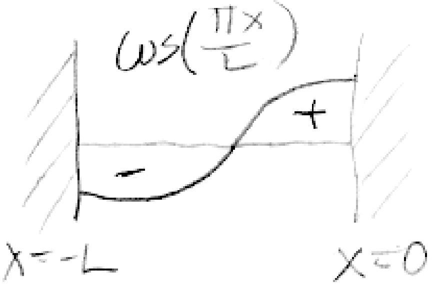
$$\boxed{\alpha_n = \sqrt{\frac{2}{L}}}$$ independent of n!

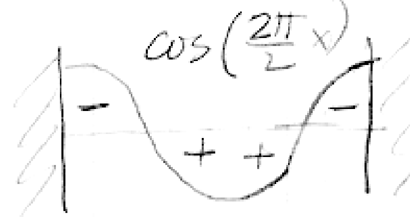
$$\boxed{\xi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L}x\right)\cos\left(\frac{\pi v}{L}t\right)}$$

what is analogy of $\hat{i}\cdot\hat{j} = 0$?

$$\int_{-L}^{0} dx\, \xi_n(x,t)\, \xi_m(x,t) \propto \frac{2}{L}\int_{-L}^{0} dx \sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m}{L}x\right)$$

$$\sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m}{L}x\right) = \frac{1}{2}\underbrace{\left(\cos\left(\frac{\pi}{L}(n+m)x\right) - \cos\left(\frac{\pi}{L}(n-m)x\right)\right)}_{\text{unless } n=m \text{, both of these integrate to } \underline{\underline{0}}}$$





so, $\displaystyle\int_{-L}^{0} dx\, \xi_n(x,t)\, \xi_m(x,t) = 0$ when $n \neq m$

like $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Just like an "arbitrary" vector $\vec{A}$ can be written $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, an "arbitrary" function $f(x)$ can be expanded in the normal modes.

$$f(x) = \sum_{n=1}^{\infty} \beta_n \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L}x\right)$$

$$\underbrace{\int_{-L}^{0} dx\, f(x)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi m}{L}\right)}_{=\beta_m} = \sum_{n=1}^{\infty} \beta_n \underbrace{\left[\frac{2}{L}\int_{-L}^{0} dx \sin\left(\frac{\pi m}{L}\right)\sin\left(\frac{\pi n}{L}\right)\right]}_{\substack{=1 \text{ when } m=n \\ =0 \text{ when } m\neq n}}$$

$$\boxed{\beta_m = \int_{-L}^{0} dx\, f(x)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi m}{L}\right)}$$

(Guitar Example)