Chapter 20  Wave Interference/Normal Modes

1. Superposition
2. Boundary Conditions
3. Standing Waves
4. Normal Modes
5. Interference
6. Resonance

1. Superposition: wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

Suppose $y_1(x,t)$ and $y_2(x,t)$ are two solutions to the wave equation.

(a) $\frac{\partial^2 y_1(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y_1(x,t)}{\partial x^2}$  
(b) $\frac{\partial^2 y_2(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y_2(x,t)}{\partial x^2}$

then: $\alpha y_1(x,t) + \beta y_2(x,t)$ is also a solution of the wave equation.

To see this, multiply (a) by $\alpha$, (b) by $\beta$, add up, factor out.

$$\alpha \frac{\partial^2 y_1}{\partial t^2} = \alpha v^2 \frac{\partial^2 y_1}{\partial x^2}$$

$$\beta \frac{\partial^2 y_2}{\partial t^2} = \beta v^2 \frac{\partial^2 y_2}{\partial x^2}$$

$$\frac{\partial^2}{\partial t^2} (\alpha y_1 + \beta y_2) = v^2 \frac{\partial^2}{\partial x^2} (\alpha y_1 + \beta y_2)$$

What use is this? To visualize boundary conditions.
string

how do we describe the effect of this... boundary condition

Try something... would \( y(x,t) = A \sin(\omega t - kx) \) work? \( \omega = \frac{2\pi}{T} \), \( k = \frac{2\pi}{\lambda} \) (this is a solution to wave equation).

Put \( x=0 \) on the wall... \( t=0 \)

\[ \begin{align*}
\text{now wait a little time.} \\
\text{moved 1/4 wavelength.} \\
+ = \frac{1}{4}T
\end{align*} \]

\[ \text{whoops! } y(0,t) \neq 0 \]

How to fix this: view the first guess as \( y_1(x,t) = A \sin(\omega t - kx) \), and add in a second solution \( y_2(x,t) \), which maintains the boundary condition of \( y(0,t) = 0 \)

\[ y_1(0,t) + y_2(0,t) = 0 \]
\[ A \sin(wt-kx) + y_2(0,t) = 0 \]

\[ y_2(0,t) = -A \sin(wt) \]

**What about spatial dependence?**

Given \( wt \), there are only two options that satisfy the wave equation:

\[ y_2(x,t) = -A \sin(wt \pm kx) \]

so the total function will be...

\[ y(x,t) = y_1(x,t) + y_2(x,t) \]

\[ = A \left( \sin(wt-kx) - \sin(wt+kx) \right) \]

\[ = 0 \quad \text{(trivial, uninteresting)} \]

\[ y(x,t) = A \left( \sin(wt-kx) - \sin(wt+kx) \right) + \text{case} \]

\[ = A \left( \sin(wt) \cos(-kx) + \cos(wt) \sin(-kx) - \sin(wt) \cos(kx) - \cos(wt) \sin(kx) \right) \]

\[ y(x,t) = -2A \sin kx \cos wt \]

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\( + = 0 \)

\[ x < 0 \]

\[ x = 0 \]
Add them together.

Another boundary condition "open end" drawn with \( \frac{\partial y}{\partial x} \neq 0 \) if so, \( F_y = -\left[ \frac{\partial y}{\partial x} \right] \) 

Tension \( F_t \), component \( F_y \), \( x \) component \( F_x \)

Frictionless Post, Massless Ring

In this case, the ring (massless) would feel infinite acceleration

\( \rightarrow \) conclude: \( \frac{\partial y}{\partial x} \bigg|_{\text{boundary}} = 0 \)

\( y(x,t) = A(\sin(wt-kx) + \sin(wt+kx)) \)

\( y(x,t) = 2A \cos(kx) \sin(wt) \)
3. Standing Waves

Imagine now boundary conditions on both ends of the string:

\[ \text{fixed} \]
\[ (\text{satisfied by superposition}) \]
\[ 2A \sin (kx) \cos \omega t \]

Need a new trick for this (\( V + \))

\[ \sin (kL) = 0 \]
\[ kL = (n) \frac{\pi}{L}, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \]

Trivial \( \Rightarrow \) sign equivalent to + sign \( \& \) time origin change.

\[ \frac{2\pi L}{\lambda} = n\pi \]
\[ n = 1, 2, 3, \ldots \]

\[ \lambda_n = \frac{2L}{n} \]
\[ n = 1, 2, 3, \ldots \]

\[ \lambda_n \cdot f_n = V \]
\[ f_n = \frac{nv}{2L} \]

String: \[ V = \sqrt{\frac{F}{\mu}} \]
\[ n = 1 \Rightarrow f_1 = \frac{v}{2L} \]
\[ \lambda_1 = 2L \]
\[ n = 2 \Rightarrow f_2 = \frac{v}{L} \]
\[ \lambda_2 = L \]
\[ n = 3 \Rightarrow f_3 = \frac{2v}{3L} \]
\[ \lambda_3 = \frac{3v}{2L} \]

Note: \( n \) = \# antinodes

This sequence of functions:
\[ s_1(x,t) = \alpha_1 \sin \left( \frac{2\pi x}{\lambda_1} \right) \cos(2\pi f_1 t) \]

Normalization constant, more later.
\[ s_1(x,t) = \alpha_1 \sin \left( \frac{\pi x}{L} \right) \cos \left( \frac{\pi v t}{L} \right) \]

\[ s_n(x,t) = \alpha_n \sin \left( \frac{\pi n x}{L} \right) \cos \left( \frac{\pi n v t}{L} \right) \]

are called the "normal modes" of the string. They are infinitely many of them. They are like "unit vectors"
Normalization

\[ \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx \]

want: \[ \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx \times 1 = 1 \]

\[ \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx \]

\[ u = \sin \left( \frac{n\pi x}{L} \right), \quad dv = dx \sin \left( \frac{n\pi x}{L} \right) \]

\[ du = \left( \frac{n\pi}{L} \right) \cos \left( \frac{n\pi x}{L} \right), \quad v = -\frac{L}{n\pi} \cos \left( \frac{n\pi x}{L} \right) \]

\[ \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = -\frac{L}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \bigg|_0^L + \int_0^L \alpha_n^2 \cos^2 \left( \frac{n\pi x}{L} \right) \, dx \]

\[ 2 \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = \int_0^L \frac{L}{2} \, dx \]

\[ \int_0^L \alpha_n^2 \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = \frac{L}{2} \]

so \[ \alpha_n^2 \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = 1 \]

\[ \alpha_n^2 \frac{L}{2} = 1 \]

\[ \alpha_n = \sqrt{\frac{2}{L}} \]

\[ \text{independent of } n \]

\[ S_n(x,t) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi t}{L} \right) \]

what is analogy of \( j = 0 \)?
\[ \int_{-L}^{L} s_n(x,) s_m(x,+) \, dx \propto \frac{2}{L} \int_{-L}^{L} \sin \left( \frac{\pi n x}{L} \right) \sin \left( \frac{\pi m x}{L} \right) \]

\[
\sin \left( \frac{\pi n x}{L} \right) \sin \left( \frac{\pi m x}{L} \right) = \frac{1}{2} \left( \cos \left( \frac{\pi}{L} (n+m) x \right) - \cos \left( \frac{\pi}{L} (n-m) x \right) \right)
\]

unless \( n = m \), both of these integrate to 0.

So, \( \int_{-L}^{L} s_n(x,) s_m(x,+) \, dx = 0 \) when \( n \neq m \)

just like \( \vec{p}_j \cdot \vec{F} = F \), \( s_j \cdot F = F \), \( s_j \cdot s = 0 \)

\[
f(x) = \sum_{n=1}^{\infty} \beta_n \sqrt{\frac{2}{L}} \sin \left( \frac{\pi n x}{L} \right)
\]

\[
\int_{-L}^{L} f(x) \sqrt{\frac{2}{L}} \sin \left( \frac{\pi m x}{L} \right) = \sum_{n=1}^{\infty} \beta_n \left[ \frac{2}{L} \int_{-L}^{L} \sin \left( \frac{\pi n x}{L} \right) \sin \left( \frac{\pi m x}{L} \right) \right]
\]

= 1 when \( m = n \)

= 0 when \( m \neq n \)

\[
\beta_m = \frac{\int_{-L}^{L} f(x) \sqrt{\frac{2}{L}} \sin \left( \frac{\pi m x}{L} \right) }{\int_{-L}^{L} f(x) \sqrt{\frac{2}{L}} \sin \left( \frac{\pi m x}{L} \right) }
\]

(Example)