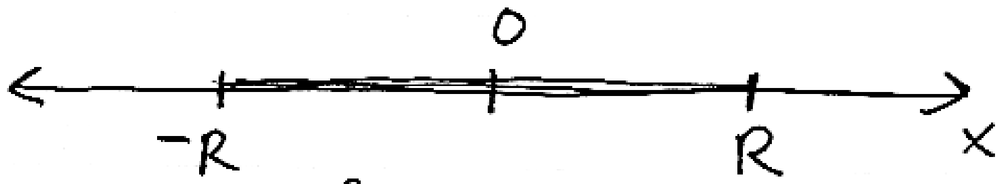
Volume, Surface Area of n-sphere

$n = 1$: "sphere" is set of points on the number line between $-R$ and R:

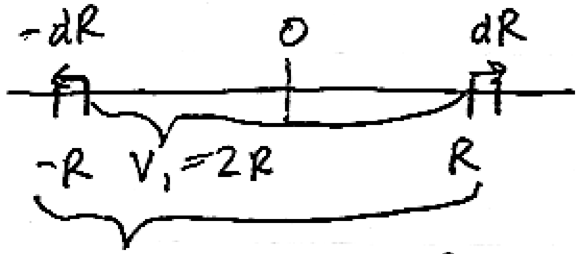


$$V_1 = \text{"volume"} = \int_{-R}^{R} dx = 2R \quad [\propto \text{length}']$$

"Surface Area" = S_1 : $S_1 \cdot dR = dV_1$

$$= 2dR$$

$$S_1 = 2 = \frac{dV_1}{dR}$$



$V_1 + dV_1 = 2R + 2dR \leftarrow dV_1 = 2dR$

This will be true generally: in n-dimensions,

$$\text{Surface Area} = S_n = \frac{dV_n}{dR}$$

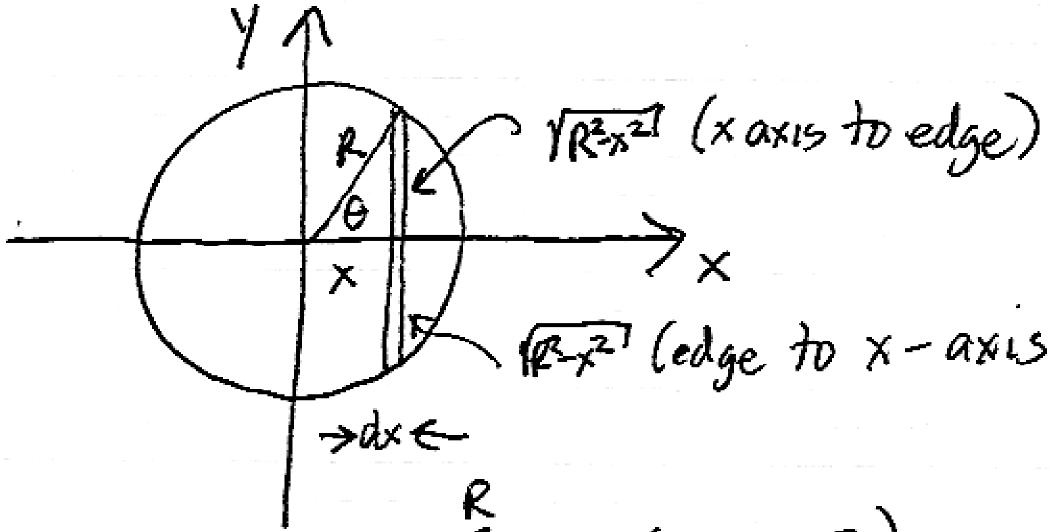
Then, again: $\boxed{\begin{aligned} V_1 &= 2R \\ S_1 &= 2 \end{aligned}}$

gauss' law

$$2 \cdot E_r = \frac{Q}{\varepsilon_0}$$

$$E_r = \frac{Q}{2\varepsilon_0} \quad [\text{like sheet}]$$

$n=2$: "sphere" is a circle, radius R



$$\text{so: } \quad V_2 = \int_{-R}^{R} dx\, V_1\left(\sqrt{R^2-x^2}\right) = 2\int_{-R}^{R} dx\,\sqrt{R^2-x^2}$$

change variables to θ:

$$\cos\theta = \frac{x}{R} \qquad \sin\theta = \frac{\sqrt{R^2-x^2}}{R}$$

$$-R\sin\theta\, d\theta = dx \qquad R\sin\theta = \sqrt{R^2-x^2}$$

$x=R$, $\theta = 0$
$x=-R$, $\theta = \pi$

$$\text{so } \quad V_2 = 2\int_{\pi}^{0} (-R\sin\theta\, d\theta)(R\sin\theta) \qquad \left\{\begin{matrix}\text{flip} \\ \text{limits} \\ \text{get} -\end{matrix}\right\}$$

$$V_2 = 2R^2 \int_0^{\pi} d\theta \sin^2\theta \quad \leftarrow \text{do by parts}$$

$$\int_0^{\pi} d\theta \sin\theta \,\sin\theta = -\underbrace{\sin\theta\cos\theta}_{0}\Big|_0^{\pi} + \int_0^{\pi} d\theta \underbrace{\cos^2\theta}_{\cos^2\theta = 1-\sin^2\theta}$$

$dv = d\theta \sin\theta$ $\quad u = \sin\theta$

$v = -\cos\theta$ $\quad du = \cancel{}\ \cos\theta\, d\theta$

$$\int_0^\pi d\theta \sin^2\theta = \int_0^\pi d\theta\,(1-\sin^2\theta)$$

$$2\int_0^\pi d\theta \sin^2\theta = \int_0^\pi d\theta = \pi$$

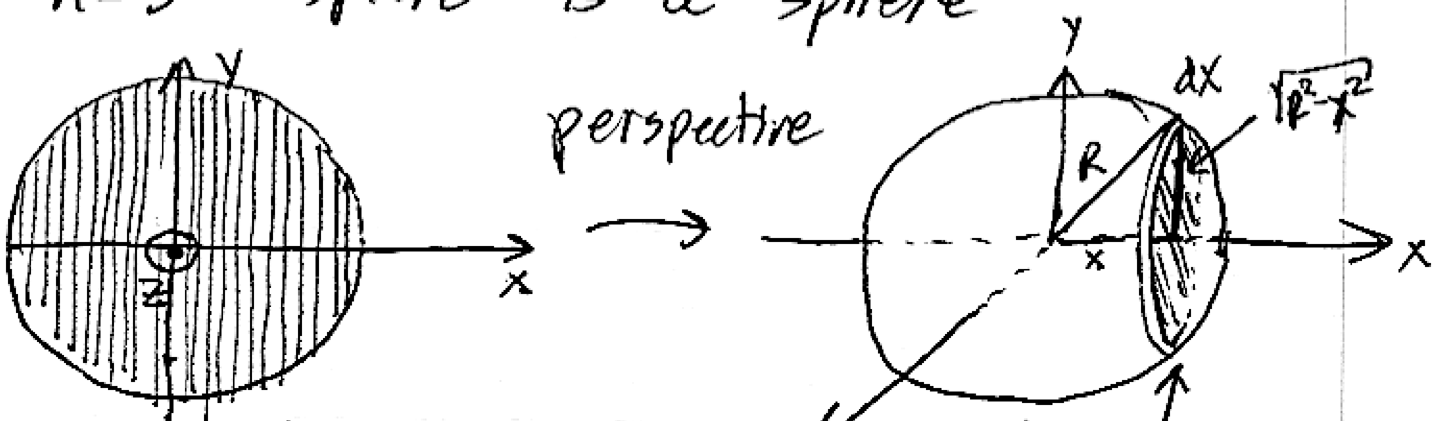
$$\int_0^\pi d\theta \sin^2\theta = \frac{\pi}{2}$$

and so $\boxed{V_2 = 2R^2 \times \frac{\pi}{2} = \pi R^2 \text{ (a.k.a. "area of a circle")}.}$

$\boxed{S_2 = \frac{dV_2}{dR} = 2\pi R \text{ (a.k.a. "circumference of a circle")}.}$

gauss:
$2\pi R \cdot E_r = \frac{Q}{\varepsilon_0}$
$E_r = \frac{Q}{2\pi\varepsilon_0 R}$

$n=3$: "sphere" is a sphere



slice up shere parallel to y-z plane...

slice volume is $dx \cdot V_2(\sqrt{R^2-x^2})$

so $V_3 = \int_{-R}^{R} dx\, V_2(\sqrt{R^2-x^2})$

$$= \pi \underbrace{\int_{-R}^{R} dx\,(R^2-x^2) = \pi\left[R^2 \cdot 2R - R^3 \cdot \frac{2}{3}\right]}$$

$\boxed{V_3 = \frac{4}{3}\pi R^3 \qquad S_3 = \frac{dV_3}{dR} = 4\pi R^2}$

gauss:
$4\pi R^2 E_r = \frac{Q}{\varepsilon_0}$
$E_r = \frac{Q}{4\pi\varepsilon_0 R^2}$

$n=4$: the key to generalizing, which has now succeeded for $n=2$, $n=3$, is:

$$V_4 = \int_{-R}^{R} dx\, V_3\left(\sqrt{R^2-x^2}\right)$$

$$= \frac{4}{3}\pi \int_{-R}^{R} dx\,(R^2-x^2)^{3/2} \leftarrow$$

do θ substitution $\cos\theta = \frac{x}{R}$

$-R\sin\theta\, d\theta = dx$

$x=R,\ \theta=0;\ x=-R,\ \theta=\pi$

$\sin\theta = \frac{\sqrt{R^2-x^2}}{R}$

$(R^2-x^2)^{3/2} = R^3\sin^3\theta$

$$V_4 = \frac{4\pi}{3}R^4 \int_0^{\pi} d\theta\, \sin^{\overset{p}{4}}\theta$$

use parts integration trick:

$$\int_0^{\pi} \underbrace{d\theta \sin\theta}_{du}\ \underbrace{\sin^{\cancel{3}\,p-1}\theta}_{v=\sin^{\cancel{3}\,p-1}\theta} = -\cos\theta\sin^{p-1}\theta\Big|_0^{\pi} + (p-1)\int_0^{\pi} d\theta\, \underbrace{\cos^2\theta}_{1-\sin^2\theta}\sin^{p-2}\theta$$

$u=-\cos\theta$ $\quad dv = (p-1)\,\cancel{\cos^{p-2}\theta}$ $\sin^{p-2}\theta\cos\theta$

$$\int_0^{\pi} d\theta \sin^p\theta = (p-1)\int_0^{\pi} d\theta \sin^{p-2}\theta - (p-1)\int_0^{\pi} d\theta\sin^p\theta$$

$$p\int_0^{\pi} d\theta\sin^p\theta = (p-1)\int_0^{\pi} d\theta\sin^{p-2}\theta$$

so
$$\int_0^{\pi} d\theta\sin^p\theta = \frac{(p-1)}{p}\cdot\int_0^{\pi} d\theta\sin^{p-2}\theta$$

so, $$\int_0^\pi d\theta \sin^4\theta = \frac{3}{4}\int_0^\pi d\theta \sin^2\theta = \frac{3}{4}\cdot\frac{\pi}{2}$$

and

$$V_4 = \frac{4\pi}{3}R^4\cdot\frac{3}{4}\frac{\pi}{2} = \frac{1}{2}\pi^2 R^4$$

$$S_4 = \frac{dV_4}{dR} = 2\pi^2 R^3$$ Gauss: $2\pi^2 R^3 E_r = \frac{Q}{\varepsilon_0}$

$$E_r = \frac{Q}{2\pi^2\varepsilon_0 R^3}$$

$n=5$

$$V_5 = \int_{-R}^{R} dx\, V_4\left(\sqrt{R^2-x^2}\right)$$

$$= \frac{1}{2}\pi^2\int_{-R}^{R} dx\,(R^2-x^2)^2 = \frac{1}{2}\pi^2 R^5\int_0^\pi \sin^5\theta\, d\theta$$

$$\int_0^\pi \sin^5\theta\, d\theta = \frac{4}{5}\int_0^\pi \sin^3\theta\, d\theta$$

$$= \frac{4\cdot2}{5\cdot3}\cdot\left[\int_0^\pi \sin\theta\, d\theta = -\cos\theta\Big|_0^\pi\right]$$

$$= \frac{4\cdot2}{5\cdot3}\cdot 2$$

so $$V_5 = \frac{1}{2}\pi^2 R^5\cdot\frac{4\cdot2}{5\cdot3}\cdot 2 = \frac{4\cdot2}{5\cdot3}\cdot\pi^2\cdot R^5$$

$$S_5 = \frac{dV_5}{dR} = \frac{4\cdot2}{3}\cdot\pi^2\cdot R^4 \qquad E_r = \frac{Q}{\varepsilon_0\cdot\frac{4\cdot2}{3}\cdot\pi^2\cdot R^4}$$

$n=6$

$$V_6 = \int_{-R}^{R} dx\, V_5\left(\sqrt{R^2-x^2}\right)$$

$$= \frac{4\cdot 2}{5\cdot 3}\cdot\pi^2\cdot R^6 \underbrace{\int_0^{\pi} d\theta \sin^6\theta}_{= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{2}}$$

$$V_6 = \frac{1}{6}\pi^3 R^6$$

$$S_6 = \frac{dV_6}{dR} = \pi^3 R^5 \qquad \text{gauss: } E_r = \frac{Q}{\pi^3\varepsilon_0 R^5}$$

You get the hang of it. An interesting way of looking at it:

$V_n = k_n R^n$ $\qquad$ $S_n = l_n R^{n-1}$

	k_n	numerically	l_n	numerically
$n=1$	2	2.0000	2	2
2	π	3.1415	2π	6.2832
3	$\frac{4\pi}{3}$	4.1887	4π	12.5664
4	$\frac{1}{2}\pi^2$	4.9348	$2\pi^2$	19.7392
5	$\frac{8}{15}\pi^2$	5.2637	$\frac{8}{3}\pi^2$	26.3190
6	$\frac{1}{6}\pi^3$	5.1677	π^3	31.0063
7	$\frac{16}{105}\pi^3$	4.7247	$\frac{16}{15}\pi^3$	33.0734
8	$\frac{1}{24}\pi^4$	4.0587	$\frac{1}{3}\pi^4$	32.4697

Note: k_n peaks for $n = 5$, l_n peaks for $n = 7$ } simply peculiar

Turns out: $n = \text{even}$, $n = 2l$

Then $$V_{2l} = \frac{\pi^l}{l!} R^{2l}$$

$n = \text{odd}$, $n = 2l+1$

Then $$V_{2l+1} = \frac{2^{2l+1}\, n!}{(2n+1)!} \pi^l R^{2l+1}$$

Relation to string theory: