MECHANICAL WAVES

Ripples on a pond, musical sounds, seismic tremors triggered by an earthquake—all these are wave phenomena. Waves can occur whenever a system is disturbed from equilibrium and when the disturbance can travel, or propagate, from one region of the system to another. As a wave propagates, it carries energy. The energy in light waves from the sun warms the surface of our planet; the energy in seismic waves can crack our planet’s crust.

This chapter and the next are about mechanical waves—waves that travel within some material called a medium. (Chapter 16 is concerned with sound, an important type of mechanical wave.) We’ll begin this chapter by deriving the basic equations for describing waves, including the important special case of sinusoidal waves in which the wave pattern is a repeating sine or cosine function. To help us understand waves in general, we’ll look at the simple case of waves that travel on a stretched string or rope.

Waves on a string play an important role in music. When a musician strums a guitar or bows a violin, she makes waves that travel in opposite directions along the instrument’s strings. What happens when these oppositely directed waves overlap is called interference. We’ll discover that sinusoidal waves can occur on a guitar or violin string only for certain special frequencies, called normal-mode frequencies, determined by the properties of the string. The normal-mode frequencies of a stringed instrument determine the pitch of the musical sounds that

When an earthquake strikes, the news of the event travels through the body of the earth in the form of seismic waves. By studying such waves, geophysicists learn about the internal structure of the earth and where future earthquake activity is likely to occur.

Some seismic waves are more energetic and destructive than others. Which aspects of a seismic wave determine how much power is carried by the wave?
the instrument produces. (In the next chapter we’ll find that interference also helps explain the pitches of wind instruments such as flutes and pipe organs.)

Not all waves are mechanical in nature. Electromagnetic waves—including light, radio waves, infrared and ultraviolet radiation, and x-rays—can propagate even in empty space, where there is no medium. We’ll explore these and other nonmechanical waves in later chapters.

15.1 | Types of Mechanical Waves

A mechanical wave is a disturbance that travels through some material or substance called the medium for the wave. As the wave travels through the medium, the particles that make up the medium undergo displacements of various kinds, depending on the nature of the wave.

Figure 15.1 shows three varieties of mechanical waves. In Fig. 15.1a the medium is a string or rope under tension. If we give the left end a small upward shake or wiggle, the wiggle travels along the length of the string. Successive sections of string go through the same motion that we gave to the end, but at successively later times. Because the displacements of the medium are perpendicular or transverse to the direction of travel of the wave along the medium, this is called a transverse wave.

In Fig. 15.1b the medium is a liquid or gas in a tube with a rigid wall at the right end and a movable piston at the left end. If we give the piston a single back-and-forth motion, displacement and pressure fluctuations travel down the length of the medium. This time the motions of the particles of the medium are back and forth along the same direction that the wave travels. We call this a longitudinal wave.
In Fig. 15.1c the medium is water in a channel, such as an irrigation ditch or canal. When we move the flat board at the left end forward and back once, a wave disturbance travels down the length of the channel. In this case the displacements of the water have both longitudinal and transverse components.

Each of these systems has an equilibrium state. For the stretched string it is the state in which the system is at rest, stretched out along a straight line. For the fluid in a tube it is a state in which the fluid is at rest with uniform pressure, and for the water in a trough it is a smooth, level water surface. In each case the wave motion is a disturbance from the equilibrium state that travels from one region of the medium to another. And in each case there are forces that tend to restore the system to its equilibrium position when it is displaced, just as the force of gravity tends to pull a pendulum toward its straight-down equilibrium position when it is displaced.

These examples have three things in common. First, in each case the disturbance travels or propagates with a definite speed through the medium. This speed is called the speed of propagation, or simply the wave speed. It is determined in each case by the mechanical properties of the medium. We will use the symbol \( v \) for wave speed. (The wave speed is not the same as the speed with which particles move when they are disturbed by the wave. We’ll return to this point in Section 15.3.) Second, the medium itself does not travel through space; its individual particles undergo back-and-forth or up-and-down motions around their equilibrium positions. The overall pattern of the wave disturbance is what travels. Third, to set any of these systems into motion, we have to put in energy by doing mechanical work on the system. The wave motion transports this energy from one region of the medium to another. Waves transport energy, but not matter, from one region to another.

### 15.2 | Periodic Waves

The transverse wave on a stretched string in Fig. 15.1a is an example of a wave pulse. The hand shakes the string up and down just once, exerting a transverse force on it as it does so. The result is a single “wiggle,” or pulse, that travels along the length of the string. The tension in the string restores its straight-line shape once the pulse has passed.

A more interesting situation develops when we give the free end of the string a repetitive, or periodic, motion. (You may want to review the discussion of periodic motion in Chapter 13 before going ahead.) Then each particle in the string also undergoes periodic motion as the wave propagates, and we have a periodic wave.

In particular, suppose we move the string up and down with simple harmonic motion (SHM) with amplitude \( A \), frequency \( f \), angular frequency \( \omega = 2\pi f \), and period \( T = 1/f = 2\pi/\omega \). Figure 15.2 shows one way to do this. The wave that results is a symmetrical sequence of crests and troughs. As we will see, periodic waves with simple harmonic motion are particularly easy to analyze; we call them sinusoidal waves. It also turns out that any periodic wave can be represented as a combination of sinusoidal waves. So this particular kind of wave motion is worth special attention.

In Fig. 15.2 the wave that advances along the string is a continuous succession of transverse sinusoidal disturbances. Figure 15.3 shows the shape of a part of the string near the left end at time intervals of \( \frac{1}{4} \) of a period, for a total time of one period. The wave shape advances steadily toward the right, as indicated by the
All particles on string oscillate in SHM with same amplitude and period.

Two particles one wavelength apart oscillate in phase with each other.

Wave travels one wavelength $\lambda$ in one period $T$.

15.3 A sinusoidal transverse wave traveling to the right along a string. The shape of the string is shown at intervals of $\frac{T}{8}$ of a period; the vertical scale is exaggerated. The blue and red dots represent particles in the string whose $x$-coordinates differ by the wavelength $\lambda$.

15.2 A block of mass $m$ attached to a spring undergoes simple harmonic motion, producing a sinusoidal wave that travels to the right on the string. The amplitude of the wave is the same as the amplitude of the spring’s oscillation. (In a real-life system a driving force would have to be applied to the block to replace the energy carried away by the wave.)

short red arrow pointing to a particular wave crest. As the wave moves, any point on the string (the blue dot, for example) oscillates up and down about its equilibrium position with simple harmonic motion. When a sinusoidal wave passes through a medium, every particle in the medium undergoes simple harmonic motion with the same frequency.

**CAUTION** Be very careful to distinguish between the motion of the transverse wave along the string and the motion of a particle of the string. The wave moves with constant speed $v$ along the length of the string, while the motion of the particle is simple harmonic and transverse (perpendicular) to the length of the string.

For a periodic wave, the shape of the string at any instant is a repeating pattern. The length of one complete wave pattern is the distance from one crest to the next, or from one trough to the next, or from any point to the corresponding point on the next repetition of the wave shape. We call this distance the wavelength of the wave, denoted by $\lambda$ (the Greek letter “lambda”). The wave pattern travels with constant speed $v$ and advances a distance of one wavelength $\lambda$ in a time interval of one period $T$. So the wave speed $v$ is given by $v = \lambda/T$, or, because $f = 1/T$,

$$v = \lambda f \quad \text{(periodic wave)} \quad (15.1)$$

The speed of propagation equals the product of wavelength and frequency. The frequency is a property of the entire periodic wave because all points on the string oscillate with the same frequency $f$.

Waves on a string propagate in just one dimension (in Fig. 15.3, along the $x$-axis). But the ideas of frequency, wavelength, and amplitude apply equally well to waves that propagate in two or three dimensions. Figure 15.4 shows a wave propagating in two dimensions on the surface of a tank of water. As with waves on a string, the wavelength is the distance from one crest to the next and the amplitude is the height of a crest.
15.2 | Periodic Waves

A series of drops falling into water produces a periodic wave that spreads radially outward. The wave crests and troughs are concentric circles. The wavelength $\lambda$ is the radial distance between adjacent crests or adjacent troughs.

In many important situations including waves on a string, the wave speed $v$ is determined entirely by the mechanical properties of the medium. In this case, increasing $f$ causes $\lambda$ to decrease so that the product $v = \lambda f$ remains the same, and waves of all frequencies propagate with the same wave speed. In this chapter we will consider only waves of this kind. (In later chapters we will study the propagation of light waves in matter for which the wave speed depends on frequency; this turns out to be the reason why prisms break white light into a spectrum and why raindrops create a rainbow.)

To understand the mechanics of a periodic longitudinal wave, we consider a long tube filled with a fluid, with a piston at the left end as in Fig. 15.1b. If we push the piston in, we compress the fluid near the piston, increasing the pressure in this region. This region then pushes against the neighboring region of fluid, and so on, and a wave pulse moves along the tube.

Now suppose we move the piston back and forth with simple harmonic motion, along a line parallel to the axis of the tube (Fig. 15.5). This motion forms regions in the fluid where the pressure and density are greater or less than the equilibrium values. We call a region of increased density a compression; a region of reduced density is a rarefaction. Figure 15.5 shows compressions as darkly shaded areas and rarefactions as lightly shaded areas. The short red arrow in Fig. 15.5 shows the position of one particular compression; the compressions and rarefactions move to the right with constant speed $v$.

The motion of a single particle of the medium, such as the one shown by a blue dot in Fig. 15.5, is simple harmonic motion parallel to the direction of wave propagation. The wavelength is the distance from one compression to the next or from one rarefaction to the next. The fundamental equation $v = \lambda f$ holds for longitudinal waves as well as for transverse waves, and indeed for all types of periodic waves. Just as for transverse waves, in this chapter and the next we will consider only situations in which the speed of longitudinal waves does not depend on the frequency.
Example 15.1  
**Wavelength of a musical sound**

Sound waves are longitudinal waves in air. The speed of sound depends on temperature; at 20°C it is 344 m/s (1130 ft/s). What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz (the approximate frequency of middle C on a piano)?

**SOLUTION**

**IDENTIFY AND SET UP:** The target variable is the wavelength λ. The wave speed \( v = 344 \text{ m/s} \) and frequency \( f = 262 \text{ Hz} \) are given, so we can use the relationship in Eq. (15.1) between \( \lambda, v, \) and \( f \) for periodic waves.

**EXECUTE:** We solve Eq. (15.1) for the target variable \( \lambda \):

\[
\lambda = \frac{v}{f} = \frac{344 \text{ m/s}}{262 \text{ Hz}} = \frac{344 \text{ m/s}}{262 \text{ s}^{-1}} = 1.31 \text{ m}
\]

Note that the units of frequency are either hertz (Hz) or inverse seconds (s\(^{-1}\)).

**EVALUATE:** What happens to the wavelength if the frequency changes? The speed of sound waves is unaffected by changes in frequency, so the relationship \( \lambda = \frac{v}{f} \) tells us that wavelength will change in inverse proportion to the frequency. As an example, the “high C” sung by coloratura sopranos is two octaves above middle C. Each octave corresponds to a factor of two in frequency, so the frequency of high C is four times that of middle C: \( f = 4(262 \text{ Hz}) = 1048 \text{ Hz} \). Hence the wavelength of high C is one-fourth as large: \( \lambda = \frac{(1.31 \text{ m})}{4} = 0.328 \text{ m} \).

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**Test Your Understanding**

If you double the wavelength of a wave on a string, what happens to the speed of the wave? What happens to the frequency of the wave?

**15.3 | Mathematical Description of a Wave**

Many characteristics of periodic waves can be described by using the concepts of wave speed, amplitude, period, frequency, and wavelength. Often, though, we need a more detailed description of the positions and motions of individual particles of the medium at particular times during wave propagation. For this description we need the concept of a wave function, a function that describes the position of any particle in the medium at any time. We will concentrate on sinusoidal waves, in which each particle undergoes simple harmonic motion about its equilibrium position.

As a specific example, let’s look at waves on a stretched string. If we ignore the sag of the string due to gravity, the equilibrium position of the string is along a straight line. We take this to be the \( x \)-axis of a coordinate system. Waves on a string are transverse; during wave motion a particle with equilibrium position \( x \) is displaced some distance \( y \) in the direction perpendicular to the \( x \)-axis. The value of \( y \) depends on which particle we are talking about (that is, \( y \) depends on \( x \) and also on the time \( t \) when we look at it). Thus \( y \) is a function of both \( x \) and \( t \); \( y = y(x,t) \). We call \( y(x,t) \) the wave function that describes the wave. If we know this function for a particular wave motion, we can use it to find the displacement (from equilibrium) of any particle at any time. From this we can find the velocity and acceleration of any particle, the shape of the string, and anything else we want to know about the behavior of the string at any time.

**Wave Function for a Sinusoidal Wave**

Let’s see how to determine the form of the wave function for a sinusoidal wave. Suppose a sinusoidal wave travels from left to right (the direction of increasing \( x \)) along the string, as in Fig. 15.3. Every particle of the string oscillates with simple
15.3 | Mathematical Description of a Wave

harmonic motion with the same amplitude and frequency. But the oscillations of particles at different points on the string are not all in step with each other. The particle marked by the blue dot in Fig. 15.3 is at its maximum positive value of \( y \) at \( t = 0 \) and returns to \( y = 0 \) at \( t = 2T/8 \); these same events occur for a particle at the center of the colored band at \( t = 4T/8 \) and \( t = 6T/8 \), exactly one half-period later. For any two particles of the string, the motion of the particle on the right (in terms of the wave, the "downstream" particle) lags behind the motion of the particle on the left by an amount proportional to the distance between the particles.

Hence the cyclic motions of various points on the string are out of step with each other by various fractions of a cycle. We call these differences phase differences, and we say that the phase of the motion is different for different points. For example, if one point has its maximum positive displacement at the same time that another has its maximum negative displacement, the two are a half-cycle out of phase. (This is the case for the blue dot in Fig. 15.3 and a point at the center of the colored band.)

Suppose that the displacement of a particle at the left end of the string \((x = 0)\), where the wave originates, is given by

\[ y(x = 0, t) = A \cos \omega t = A \cos 2\pi ft \quad (15.2) \]

That is, the particle oscillates in simple harmonic motion with amplitude \( A \), frequency \( f \), and angular frequency \( \omega = 2\pi f \). The notation \( y(x = 0, t) \) reminds us that the motion of this particle is a special case of the wave function \( y(x, t) \) that describes the entire wave. At \( t = 0 \) the particle at \( x = 0 \) is at its maximum positive displacement \((y = A)\) and is instantaneously at rest (because the value of \( y \) is a maximum).

The wave disturbance travels from \( x = 0 \) to some point \( x \) to the right of the origin in an amount of time given by \( x/v \), where \( v \) is the wave speed. So the motion of point \( x \) at time \( t \) is the same as the motion of point \( x = 0 \) at the earlier time \( t - x/v \). Hence we can find the displacement of point \( x \) at time \( t \) by simply replacing \( t \) in Eq. (15.2) by \( (t - x/v) \). When we do that, we find the following expression for the wave function:

\[ y(x, t) = A \cos [\omega(t - x/v)] \]

Because \( \cos(-\theta) = \cos \theta \), we can rewrite the wave function as

\[ y(x, t) = A \cos \left[ \omega \left( \frac{x}{v} - t \right) \right] = A \cos 2\pi f \left( \frac{x}{v} - t \right) \quad (15.3) \]

(sinusoidal wave moving in \( +x \)-direction)

The displacement \( y(x, t) \) is a function of both the location \( x \) of the point and the time \( t \). We could make Eq. (15.3) more general by allowing for different values of the phase angle, as we did for simple harmonic motion in Section 13.2, but for now we omit this.

We can rewrite the wave function given by Eq. (15.3) in several different but useful forms. We can express it in terms of the period \( T = 1/f \) and the wavelength \( \lambda = v/f \):

\[ y(x, t) = A \cos 2\pi \left( \frac{x}{\lambda} - \frac{t}{T} \right) \quad (15.4) \]

(sinusoidal wave moving in \( +x \)-direction)
We get another convenient form of the wave function if we define a quantity \( k \), called the \textbf{wave number}:

\[
k = \frac{2\pi}{\lambda} \quad \text{(wave number)}
\]  

Substituting \( \lambda = 2\pi/k \) and \( f = \omega/2\pi \) into the wavelength-frequency relation \( v = \lambda f \) gives

\[
\omega = vk \quad \text{(periodic wave)}
\]  

We can then rewrite Eq. (15.4) as

\[
y(x, t) = A \cos(kx - \omega t)
\]  

(sinusoidal wave moving in \(+x\)-direction)

Which of these various forms for the wave function \( y(x, t) \) we use in any specific problem is a matter of convenience. Note that \( \omega \) has units rad/s, so for unit consistency in Eqs. (15.6) and (15.7) the wave number \( k \) must have the units rad/m. (Some physicists define the wave number as \( 1/\lambda \) rather than \( 2\pi/\lambda \). When reading other texts, be sure to determine how this term is defined.)

The wave function \( y(x, t) \) is graphed as a function of \( x \) for a specific time \( t \) in Fig. 15.6a. This graph gives the displacement \( y \) of a particle from its equilibrium position as a function of the coordinate \( x \) of the particle. If the wave is a transverse wave on a string, the graph in Fig. 15.6a represents the shape of the string at that instant, like a flash photograph of the string. In particular, at time \( t = 0 \),

\[
y(x, t = 0) = A \cos kx = A \cos \frac{2\pi x}{\lambda}
\]

Figure 15.6b is a graph of the wave function versus time \( t \) for a specific coordinate \( x \). This graph gives the displacement \( y \) of the particle at that coordinate as a function of time—that is, it describes the motion of that particle. In particular, at the position \( x = 0 \),

\[
y(x = 0, t) = A \cos(-\omega t) = A \cos \omega t = A \cos \frac{2\pi t}{T}
\]

This is consistent with our original statement about the motion at \( x = 0 \), Eq. (15.2).

\textbf{CAUTION} Although they may look the same at first glance, Figs. 15.6a and 15.6b are not identical. Figure 15.6a is a picture of the shape of the string at \( t = 0 \), while Fig. 15.6b is a graph of the displacement \( y \) of a particle at \( x = 0 \) as a function of time.
We can modify Eqs. (15.3) through (15.7) to represent a wave traveling in the negative $x$-direction. In this case the displacement of point $x$ at time $t$ is the same as the motion of point $x = 0$ at the later time $(t + x/v)$. So in Eq. (15.2) we replace $t$ by $(t + x/v)$. For a wave traveling in the negative $x$-direction,

$$y(x, t) = A \cos \left( \frac{x}{v} + t \right) = A \cos \left( \frac{x}{A} + \frac{t}{T} \right) = A \cos (kx + \omega t)$$

(sinuousoidal wave moving in $-x$-direction) \hspace{1cm} (15.8)

In the expression $y(x, t) = A \cos (kx \pm \omega t)$ for a wave traveling in the $-x$ or $+x$-direction, the quantity $(kx \pm \omega t)$ is called the phase. It plays the role of an angular quantity (always measured in radians) in Eq. (15.7) or (15.8), and its value for any values of $x$ and $t$ determines what part of the sinusoidal cycle is occurring at a particular point and time. For a crest (where $y = A$ and the cosine function has the value 1), the phase could be 0, 2$\pi$, 4$\pi$, and so on; for a trough (where $y = -A$ and the cosine has the value -1) it could be $\pi$, 3$\pi$, 5$\pi$, and so on.

The wave speed is the speed with which we have to move along with the wave to keep alongside a point of a given phase, such as a particular crest of a wave on a string. For a wave traveling in the $+x$-direction, that means $kx - \omega t = $ constant. Taking the derivative with respect to $t$, we find $k \frac{dx}{dt} = \omega$, or

$$\frac{dx}{dt} = \frac{\omega}{k}$$

Comparing this with Eq. (15.6), we see that $dx/dt$ is equal to the speed $v$ of the wave. Because of this relationship, $v$ is sometimes called the phase velocity of the wave. (Phase speed would be a better term.)

**Problem-Solving Strategy**

**Mechanical Waves**

**IDENTIFY the relevant concepts:** Wave problems fall into two broad categories. Kinematics problems are concerned with describing wave motion; they involve wave speed $v$, wavelength $\lambda$ (or wave number $k$), frequency $f$ (or angular frequency $\omega$), and amplitude $A$. They may also involve the position, velocity, and acceleration of individual particles in the medium. Dynamics problems also use concepts from Newton’s laws such as force and mass. As an example, later in this chapter we’ll encounter problems that involve the relation of wave speed to the mechanical properties of the wave medium. We’ll get into these relations in the next section and in Chapter 16.

As always, make sure that you identify the target variable(s) for the problem. In some cases it will be the wavelength, frequency, or wave speed; in other cases you’ll be asked to find an expression for the wave function.

**SET UP the problem using the following steps:**

1. Make a list of the quantities whose values are given. To help you visualize the situation, you’ll find it useful to sketch graphs of $y$ versus $x$ (like Fig. 15.6a) and of $y$ versus $t$ (like Fig. 15.6b). Label your graphs with the values of the known quantities.

2. Decide which equations you’ll need to use. If any two of $v$, $f$, and $\lambda$ are given, you’ll need to use Eq. (15.1) ($v = \lambda f$) to find the third quantity (see Example 15.1). If the problem involves the angular frequency $\omega$ and/or the wave number $k$, you’ll need to use the definitions of those quantities and Eq. (15.6) ($\omega = vk$). You may also need the various forms of the wave function given in Eqs. (15.3), (15.4), and (15.7).

3. If the wave speed isn’t given, and you don’t have enough information to determine it using $v = \lambda f$, you may be able to find $v$ using the relationship between $v$ and the mechanical properties of the system. (In the next section we’ll develop this relation for waves on a string.)

**EXECUTE the solution as follows:**

Solve for the unknown quantities using the equations you’ve selected. In some problems all you need to do is find the value of one of the wave variables.
If you're asked to determine the wave function, you need to know \( A \) and any two of \( u \), \( A \), and \( f \) or \( u \), \( f \), and \( \omega \). Once you have this information, you can use it in Eq. (15.3), (15.4), or (15.7) to get the specific wave function for the problem at hand. Once you have that, you can find the value of \( y \) at any point (value of \( x \)) and at any time by substituting into the wave function.

**EVALUATE your answer:** Look at your results with a critical eye. Check to see whether the values of \( u \), \( f \), and \( \lambda \) (or \( v \), \( \omega \), and \( k \)) agree with the relationships given in Eq. (15.1) or (15.6). If you've calculated the wave function, check one or more special cases for which you can guess what the results ought to be.

---

**Example 15.2 Wave on a clothesline**

Your cousin Throckmorton is playing with the clothesline. He unties one end, holds it taut, and wiggles the end up and down sinusoidally with frequency 2.00 Hz and amplitude 0.075 m. The wave speed is \( v = 12.0 \) m/s. At time \( t = 0 \) the end has maximum positive displacement and is instantaneously at rest. Assume no wave bounces back from the far end to muddle up the pattern. a) Find the amplitude, angular frequency, period, wavelength, and wave number of the wave. b) Write a wave function describing the wave. c) Write equations for the displacement as a function of time of Throckmorton's end of the clothesline and of a point 3.00 m from his end.

**IDENTIFY:** This is a kinematics problem about the motion of the clothesline. Since Throcky moves his hand in a sinusoidal way, he produces a sinusoidal wave that propagates down the clothesline. Hence we can use all of the expressions we've developed in this section. Our target variables in part (a) are amplitude \( A \), angular frequency \( \omega \), period \( T \), wavelength \( \lambda \), and wave number \( k \), so we need to use the equations that relate these quantities. In parts (b) and (c) our target “variables” are actually expressions for displacement; to find these, we use the general equations for the wave function of a sinusoidal wave.

**SET UP:** A photograph of the clothesline at time \( t = 0 \) would look just like Fig. 15.6a, with the maximum displacement at \( x = 0 \) (the end that Throcky has in his hand). We take the positive \( x \)-direction to be the direction in which the wave propagates, so we can use Eqs. (15.4) and (15.7) to describe the displacement of the clothesline as a function of position \( x \) and time \( t \). We also use the relationships \( f = 1/T \), \( \omega = 2\pi f \), \( k = 2\pi/\lambda \), \( v = \lambda f \), and \( \omega = \omega k \).

**EXECUTE:** a) The amplitude \( A \) of the wave is just the amplitude of the motion of the end of the clothesline, \( A = 0.075 \) m. Similarly, the wave frequency is \( f = 2.00 \) Hz, the same as the frequency of the end of the clothesline. The angular frequency is

\[
\omega = 2\pi f = \left( 2\pi \text{ rad/cycle} \right)\left( 2.00 \text{ cycles/s} \right) = 4.00\pi \text{ rad/s}
\]

The period is \( T = 1/f = 0.500 \) s. We get the wavelength from Eq. (15.1):

\[
\lambda = \frac{v}{f} = \frac{12.0 \text{ m/s}}{2.00 \text{ s}^{-1}} = 6.00 \text{ m}
\]

We find the wave number from Eq. (15.5) or (15.6):

\[
k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{6.00 \text{ m}} = 1.05 \text{ rad/m}
\]

b) Since we found the values of \( A \), \( T \), and \( \lambda \) in part (a), we can write down the wave function using Eq. (15.4):

\[
y(x, t) = A \cos 2\pi \left( \frac{x}{\lambda} - \frac{t}{T} \right) = (0.075 \text{ m}) \cos 2\pi \left( \frac{x}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}} \right)
\]

We can also get this same equation from Eq. (15.7) by using the values of \( \omega \) and \( k \) we obtained in part (a).

c) With our choice of the positive \( x \)-direction, the two points in question are at \( x = 0 \) and \( x = +3.00 \) m. For each point, we can find the displacement as a function of time by substituting these values of \( x \) into the wave function we found in part (b):

\[
y(x = 0, t) = (0.075 \text{ m}) \cos 2\pi \left( \frac{0}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}} \right) = (0.075 \text{ m}) \cos (12.6 \text{ rad/s})t
\]

\[
y(x = +3.00 \text{ m}, t) = (0.075 \text{ m}) \cos 2\pi \left( \frac{3.00 \text{ m}}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}} \right) = (0.075 \text{ m}) \cos (12.6 \text{ rad/s})t
\]

**EVALUATE:** In part (b), the quantity \( (1.05 \text{ rad/m})x - (12.6 \text{ rad/s})t \) is the phase of a point \( x \) on the string at time \( t \). The phases of the two points in part (c) differ by \( \pi \) because these points are separated by one half-wavelength \( \lambda/2 = (6.00 \text{ m})/2 = 3.00 \text{ m} \). Both points oscillate in SHM with the same frequency and amplitude, but their
15.3 | Mathematical Description of a Wave

oscillations are one half-cycle out of phase. Thus, while a graph of \( y \) versus \( t \) for the point at \( x = 0 \) is a cosine curve (like Fig. 15.6b), a graph of \( y \) versus \( t \) for the point \( x = 3.00 \) m is a negative cosine (the same as a cosine curve shifted by one half-cycle).

Using the expression for \( y(x = 0, t) \) in part (c), can you show that the end of the string at \( x = 0 \) is instantaneously at rest at \( t = 0 \), just as we stated at the beginning of this example? (Hint: calculate the velocity at this point by taking the derivative of \( y \) with respect to \( t \)).

**Particle Velocity and Acceleration in a Sinusoidal Wave**

From the wave function we can get an expression for the transverse velocity of any particle in a transverse wave. We call this \( v \), to distinguish it from the wave propagation speed \( v \).

To find the transverse velocity \( v \), at a particular point \( x \), we take the derivative of the wave function \( y(x, t) \) with respect to \( t \), keeping \( x \) constant. If the wave function is

\[
y(x, t) = A \cos (kx - \omega t)
\]

then

\[
v(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin (kx - \omega t)
\]

(15.9)

The \( \partial \) in this expression is a modified \( d \), used to remind us that \( y(x, t) \) is a function of two variables and that we are allowing only one \( (t) \) to vary. The other \( (x) \) is constant because we are looking at a particular point on the string. This derivative is called a partial derivative. If you haven’t reached this point yet in your study of calculus, don’t fret; it’s a simple idea.

Equation (15.9) shows that the transverse velocity of a particle varies with time, as we expect for simple harmonic motion. The maximum particle speed is \( \omega A \); this can be greater than, less than, or equal to the wave speed \( v \), depending on the amplitude and frequency of the wave.

The acceleration of any particle is the second partial derivative of \( y(x, t) \) with respect to \( t \):

\[
a(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \cos (kx - \omega t) = -\omega^2 y(x, t)
\]

(15.10)

The acceleration of a particle equals \( -\omega^2 \) times its displacement, which is the result we obtained in Section 13.2 for simple harmonic motion.

We can also compute partial derivatives of \( y(x, t) \) with respect to \( x \), holding \( t \) constant. This corresponds to studying the shape of the string at one instant of time, like a flash photo. The first derivative \( \partial y(x, t)/\partial x \) is the slope of the string at any point. The second partial derivative with respect to \( x \) is the curvature of the string:

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 A \cos (kx - \omega t) = -k^2 y(x, t)
\]

(15.11)

From Eqs. (15.10) and (15.11) and the relation \( \omega = \nu k \) we see that

\[
\frac{\partial^2 y(x, t)/\partial t^2}{\partial^2 y(x, t)/\partial x^2} = \frac{\omega^2}{k^2} = \nu^2
\]

and

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 y(x, t)}{\partial t^2}
\]

(wave equation)

(15.12)
We've derived Eq. (15.12) for a wave traveling in the positive x-direction. You can use the same steps to show that the wave function for a sinusoidal wave propagating in the negative x-direction, \( y(x, t) = A \cos (kx + wt) \), also satisfies this equation.

Equation (15.12), called the wave equation, is one of the most important equations in all of physics. Whenever it occurs, we know that a disturbance can propagate as a wave along the x-axis with wave speed \( v \). The disturbance need not be a sinusoidal wave; we'll see in the next section that any wave on a string obeys Eq. (15.12), whether the wave is periodic or not (see also Problems 15.54 and 15.59). In Chapter 32 we will find that electric and magnetic fields satisfy the wave equation; the wave speed will turn out to be the speed of light, which will lead us to the conclusion that light is an electromagnetic wave.

Figure 15.7a shows the velocity \( v_x \) and acceleration \( a_x \), given by Eqs. (15.9) and (15.10), for several points on a string as a sinusoidal wave passes along it. Note that at points where the string has an upward curvature \( (\frac{\partial^2 y}{\partial x^2} > 0) \), the acceleration of that point is positive \( (a_x = \frac{\partial^2 y}{\partial t^2} > 0) \); this follows from the wave equation, Eq. (15.12). For the same reason the acceleration is negative \( (a_x = \frac{\partial^2 y}{\partial t^2} < 0) \) at points where the string has a downward curvature \( (\frac{\partial^2 y}{\partial x^2} < 0) \), and the acceleration is zero \( (a_x = \frac{\partial^2 y}{\partial t^2} = 0) \) at points of inflection where the curvature is zero \( (\frac{\partial^2 y}{\partial x^2} = 0) \). We emphasize again that \( v_x \) and \( a_x \) are the transverse velocity and acceleration of points on the string; these points move along the y-direction, not along the propagation direction of the wave. Figure 15.7b shows the transverse motions of several points on the string.

The concept of wave function is equally useful with longitudinal waves, and everything we have said about wave functions can be adapted to this case. The quantity \( y \) still measures the displacement of a particle of the medium from its equilibrium position; the difference is that for a longitudinal wave, this displacement is parallel to the x-axis instead of perpendicular to it. We'll discuss longitudinal waves in detail in Chapter 16.

15.7 (a) Another view of the wave at \( t = 0 \) in Fig. 15.6a. The vectors show the transverse velocity \( v_x \) and transverse acceleration \( a_x \) at several points on the string. (b) The two curves show the wave at \( t = 0 \) and \( t = 0.05T \). During this interval, a particle at point 1 is displaced to point 1', a particle at point 2 is displaced to point 2', and so on.
The wave function for a sinusoidal wave is

\[ y(x, t) = (2.50 \text{ mm}) \cos \left( (2.00 \text{ m}^{-1})x + (3.20 \text{ rad/s})t \right) \]

In which direction is this wave propagating? What is its wave number? Its angular frequency? Its amplitude?

15.4 | **Speed of a Transverse Wave**

One of the key properties of any wave is the wave speed. Light waves in air have a much greater speed of propagation than do sound waves in air (3.00 × 10⁸ m/s versus 344 m/s); that’s why you see the flash from a bolt of lightning before you hear the clap of thunder. In this section we’ll see what determines the speed of propagation of one particular kind of wave: transverse waves on a string. The speed of these waves is important to understand in its own right because it is an essential part of analyzing stringed musical instruments, as we’ll discuss later in this chapter. Furthermore, the speeds of many kinds of mechanical waves turn out to have the same basic mathematical expression as does the speed of waves on a string.

The physical quantities that determine the speed of transverse waves on a string are the tension in the string and its mass per unit length (also called linear mass density). We might guess that increasing the tension should increase the restoring forces that tend to straighten the string when it is disturbed, thus increasing the wave speed. We might also guess that increasing the mass should make the motion more sluggish and decrease the speed. Both these guesses turn out to be right. We’ll develop the exact relationship between wave speed, tension, and mass per unit length by two different methods. The first is simple in concept and considers a specific wave shape; the second is more general but also more formal. Choose whichever you like better.

**Wave Speed on a String: First Method**

We consider a perfectly flexible string (Fig. 15.8). In the equilibrium position the tension is \( F \), and the linear mass density (mass per unit length) is \( \mu \). (When portions of the string are displaced from equilibrium, the mass per unit length decreases a little, and the tension increases a little.) We ignore the weight of the string so that when the string is at rest in the equilibrium position, the string forms a perfectly straight line as in Fig. 15.8a.

Starting at time \( t = 0 \), we apply a constant upward force \( F \) at the left end of the string. We might expect that the end would move with constant acceleration; that would happen if the force were applied to a point mass. But here the effect of the force \( F \) is to set successively more and more mass in motion. The wave travels with constant speed \( v \), so the division point \( P \) between moving and nonmoving portions moves with the same constant speed \( v \) (Fig. 15.8b).

Figure 15.8b shows that all particles in the moving portion of the string move upward with constant velocity \( v_z \), not constant acceleration. To see why this is so, we note that the impulse of the force \( F \) up to time \( t \) is \( F \cdot t \). According to the impulse-momentum theorem (Section 8.1), the impulse is equal to the change in the total transverse component of momentum \( (mv_z - 0) \) of the moving part of
Mechanical Waves

The total momentum thus must increase proportionately with time. But since the division point $P$ moves with constant speed, the length of string that is in motion and hence the total mass $m$ in motion are also proportional to the time $t$ that the force has been acting. So the change of momentum must be associated entirely with the increasing amount of mass in motion, not with an increasing velocity of an individual mass element. That is, $mv_y$ changes because $m$, not $v_y$, changes.

At time $t$, the left end of the string has moved up a distance $v_y t$, and the boundary point $P$ has advanced a distance $vt$. The total force at the left end of the string has components $F$ and $F_x$. Why $F$? There is no motion in the direction along the length of the string, so there is no unbalanced horizontal force. Therefore $F$, the magnitude of the horizontal component, does not change when the string is displaced. In the displaced position the tension is $(F^2 + F_x^2)^{1/2}$ (greater than $F$), and the string stretches somewhat.

To derive an expression for the wave speed $v$, we again apply the impulse-momentum theorem to the portion of the string in motion at time $t$, that is, the portion to the left of $P$ in Fig. 15.8b. The transverse impulse (transverse force times time) is equal to the change of transverse momentum of the moving portion (mass times transverse component of velocity). The impulse of the transverse force $F_x$ in time $t$ is $F_x t$. In Fig. 15.8b the right triangle whose vertex is at $P$, with sides $v_y t$ and $v t$, is similar to the right triangle whose vertex is at the position of the hand, with sides $F_y$ and $F$. Hence

$$\frac{F_y}{F} = \frac{v_y t}{vt} \quad F_y = F \frac{v_y}{v}$$

and

$$\text{Transverse impulse} = F_y t = F \frac{v_y}{v} t$$
The mass of the moving portion of the string is the product of the mass per unit length \( \mu \) and the length \( vt \), or \( \mu vt \). The transverse momentum is the product of this mass and the transverse velocity \( v_y \):

\[
\text{Transverse momentum} = (\mu vt)v_y
\]

We note again that the momentum increases with time not because mass is moving faster, as was usually the case in Chapter 8, but because more mass is brought into motion. But the impulse of the force \( F_y \) is still equal to the total change in momentum of the system. Applying this relation, we obtain

\[
\frac{F_y}{t} = \mu v_y
\]

Solving this for \( v_y \), we find

\[
v = \sqrt{\frac{F}{\mu}} \quad \text{(speed of a transverse wave on a string)} \quad (15.13)
\]

Equation (15.13) confirms our prediction that the wave speed \( v \) should increase when the tension \( F \) increases but decrease when the mass per unit length \( \mu \) increases (Fig. 15.9).

Note that \( v_y \) does not appear in Eq. (15.13); thus the wave speed doesn’t depend on \( v_y \). Our calculation considered only a very special kind of pulse, but we can consider any shape of wave disturbance as a series of pulses with different values of \( v_y \). So even though we derived Eq. (15.13) for a special case, it is valid for any transverse wave motion on a string, including the sinusoidal and other periodic waves we discussed in Section 15.3. Note also that the wave speed doesn’t depend on the amplitude or frequency of the wave, in accordance with our assumptions in Section 15.3.

**Wave Speed on a String: Second Method**

Here is an alternative derivation of Eq. (15.13). If you aren’t comfortable with partial derivatives, it can be omitted. We apply Newton’s second law, \( \sum \vec{F} = m\vec{a} \), to a small segment of string whose length in the equilibrium position is \( \Delta x \) (Fig. 15.10). The mass of the segment is \( m = \mu \Delta x \); the forces at the ends are represented in terms of their \( x \)- and \( y \)-components. The \( x \)-components have equal magnitude \( F \) and add to zero because the motion is transverse and there is no component of acceleration in the \( x \)-direction. To obtain \( F_{1x} \) and \( F_{2x} \), we note that the ratio \( F_{1y}/F \) is

15.4  |  Speed of a Transverse Wave

15.9 These cables have a relatively large amount of mass per unit length \( (\mu) \) and a low tension \( (F) \). If the cables are disturbed—say, by a bird landing on them—transverse waves will travel along them at a slow speed \( v = \sqrt{F/\mu} \).
equal in magnitude to the slope of the string at point \( x \) and that \( F_2/F \) is equal to the slope at point \( x + \Delta x \). Taking proper account of signs, we find

\[
\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \quad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}
\]  

(15.14)

The notation reminds us that the derivatives are evaluated at points \( x \) and \( x + \Delta x \), respectively. From Eq. (15.14) we find that the net \( y \)-component of force is

\[
F_y = F_{1y} + F_{2y} = F \left[ \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right]
\]  

(15.15)

We now equate \( F \), from Eq. (15.15) to the mass \( \mu \Delta x \) times the \( y \)-component of acceleration \( \partial^2 y/\partial t^2 \). We obtain

\[
F \left[ \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right] = \mu \Delta x \frac{\partial^2 y}{\partial t^2}
\]  

(15.16)

or, dividing by \( F \Delta x \),

\[
\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu \partial^2 y}{F \partial t^2}
\]

(15.17)

We now take the limit as \( \Delta x \to 0 \). In this limit, the left side of Eq. (15.17) becomes the derivative of \( \partial y/\partial x \) with respect to \( x \) (at constant \( t \)), that is, the second (partial) derivative of \( y \) with respect to \( x \):

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\mu \partial^2 y}{F \partial t^2}
\]

(15.18)

Now, finally, comes the punch line of our story. Equation (15.18) has exactly the same form as the wave equation, Eq. (15.12), that we derived at the end of Section 15.3. That equation and Eq. (15.18) describe the very same wave motion, so they must be identical. Comparing the two equations, we see that for this to be so, we must have

\[
v = \sqrt{\frac{F}{\mu}}
\]

(15.19)

which is the same expression as Eq. (15.13).

In going through this derivation, we didn’t make any special assumptions about the shape of the wave. Since our derivation led us to rediscover Eq. (15.12), the wave equation, we conclude that the wave equation is valid for waves on a string that have any shape.

Equation (15.13) or (15.19) gives the wave speed only for the special case of mechanical waves on a stretched string or rope. Remarkably, it turns out that for many types of mechanical waves, including waves on a string, the expression for wave speed has the same general form:

\[
v = \sqrt{\frac{(\text{restoring force returning the system to equilibrium})}{(\text{inertia resisting the return to equilibrium})}}
\]

To interpret this expression, let’s look at the now-familiar case of waves on a string. The tension \( F \) in the string plays the role of the restoring force; it tends to
bring the string back to its undisturbed, equilibrium configuration. The mass of the string—or, more properly, the linear mass density $\mu$—provides the inertia that prevents the string from returning instantaneously to equilibrium. Hence we have $v = \sqrt{F/\mu}$ for the speed of waves on a string.

In Chapter 16 we’ll see a similar expression for the speed of sound waves in a gas. Roughly speaking, the gas pressure provides the force that tends to return the gas to its undisturbed state when a sound wave passes through. The inertia is provided by the density, or mass per unit volume, of the gas.

### Example 15.3  Varying the speed of a wave

In Example 15.2 the linear mass density of the clothesline is 0.250 kg/m. a) How much tension does Throcky have to apply to produce the observed wave speed of 12.0 m/s? b) If the tension is increased to four times the value in (a) but the frequency is still 2.00 Hz, what will be the wavelength of the wave on the clothesline?

#### SOLUTION

**IDENTIFY:** The target variable is the tension in part (a) and the wavelength in part (b). Both parts of the problem involve dynamics, that is, the relationship between the wave speed and the properties of the clothesline itself (tension and linear mass density). Part (b) also involves kinematics, since we’ll need the relationship between the wave speed, wavelength, and frequency.

**SET UP:** In part (a) we’re given the linear mass density $\mu$ and the wave speed $v$. Hence we can find the tension $F$ using Eq. (15.13), $v = \sqrt{F/\mu}$. In part (b) we use this same relationship to find the new wave speed with the new value of tension, then find the new wavelength $\lambda$ using Eq. (15.1), $\lambda = v/f$.

**EXECUTE:** (a) Solving Eq. (15.13) for $F$, we find

$$F = \mu v^2 = (0.250 \text{ kg/m}) (12.0 \text{ m/s})^2 = 36.0 \text{ kg}\cdot\text{m/s}^2$$

$$= 36.0 \text{ N} = 8.09 \text{ lb}$$

which is well within Throcky’s capabilities.

b) Equation (15.13) says that the wave speed is proportional to the square root of the tension $F$. Furthermore, Eq. (15.1) tells us that $\lambda = v/f$, so that the wavelength is linearly proportional to the wave speed (for constant frequency $f$, which is the case in this problem). So if we increase the value of $F$ by a factor of 4, the wave speed and the wavelength both increase by a factor of $\sqrt{4} = 2$. Hence, using the old wavelength value from Example 15.2 will give the new wavelength

$$\lambda = 2(6.00 \text{ m}) = 12.0 \text{ m}$$

**EVALUATE:** We can check our answer to part (b) by using the new tension $F = 4(36.0 \text{ N})$ in the expression $v = \sqrt{F/\mu}$ to find the new wave speed, then calculating the new wavelength using $\lambda = v/f$:

$$v = \sqrt{\frac{144 \text{ N}}{0.250 \text{ kg/m}}} = 24.0 \text{ m/s} \quad \text{so}$$

$$\lambda = \frac{24.0 \text{ m/s}}{2.00 \text{ s}^{-1}} = 12.0 \text{ m}$$

### Example 15.4  Calculating wave speed

One end of a nylon rope is tied to a stationary support at the top of a vertical mine shaft 80.0 m deep (Fig. 15.11). The rope is stretched taut by a box of mineral samples with mass 20.0 kg attached at the lower end. The mass of the rope is 2.00 kg. The geologist at the bottom of the mine signals to his colleague at the top by jerking the rope sideways. a) What is the speed of a transverse wave on the rope? b) If a point on the rope is given a transverse simple harmonic motion with a frequency of 2.00 Hz, how many cycles of the wave are there in the rope’s length?

#### SOLUTION

**IDENTIFY:** This example is similar to the previous one; it involves both dynamics [in part (a)] and kinematics [in part (b)].

In part (a) the target variable is the wave speed. The new feature is that the tension is provided by the weight of the box of samples. In fact, the weight of the rope itself contributes to the tension, which means that the tension is different at the top and bottom of the rope. We’ll ignore this effect here since the weight of the rope is small compared to the weight of the samples.
Note that the target variable in part (b) is actually the number of wavelengths that fit into the length of the rope.

**SET UP:** As in Example 15.3, we use the relationship \( v = \sqrt{F/\mu} \) in part (a). If we neglect the weight of the rope itself, the tension \( F \) is just equal to the weight of the box. In part (b) we use the equation \( v = fA \) to find the wavelength, which we then compare to the 80.0-m length of the rope.

**EXECUTE:**

a) The tension in the rope (due to the sample box) is

\[
F = m_{\text{samples}}g = (20.0 \text{ kg})(9.80 \text{ m/s}^2) = 196 \text{ N}
\]

and the mass per unit length of the rope is

\[
\mu = \frac{m_{\text{rope}}}{L} = \frac{2.00 \text{ kg}}{80.0 \text{ m}} = 0.0250 \text{ kg/m}
\]

Hence, from Eq. (15.13), the wave speed is

\[
v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{196 \text{ N}}{0.0250 \text{ kg/m}}} = 88.5 \text{ m/s}
\]

b) From Eq. (15.1),

\[
\lambda = \frac{v}{f} = \frac{88.5 \text{ m/s}}{2.00 \text{ s}^{-1}} = 44.3 \text{ m}
\]

The length of the rope is 80.0 m, so the number of wave cycles in the rope is

\[
\frac{80.0 \text{ m/s}}{44.3 \text{ m/cycle}} = 1.81 \text{ cycles}
\]

**EVALUATE:** If we account for the weight of the rope, the tension is greater at the top of the rope than at the bottom. Hence the wave speed increases and the wavelength decreases as the wave travels up the rope. Can you verify that the wave speed at the top is 92.9 m/s?

---

**Test Your Understanding**

The six strings of a guitar are of the same length and are under nearly the same tension, but have different thicknesses. On which string do waves travel the fastest?

---

**15.5 Energy in Wave Motion**

Every wave motion has energy associated with it. The energy we receive from sunlight and the destructive effects of ocean surf and earthquakes bear this out. To produce any of the wave motions we have discussed in this chapter, we have to apply a force to a portion of the wave medium; the point where the force is applied moves, so we do work on the system. As the wave propagates, each portion of the medium exerts a force and does work on the adjoining portion. In this way a wave can transport energy from one region of space to another.

As an example of energy considerations in wave motion, let's look again at transverse waves on a string. How is energy transferred from one portion of string to another? Picture a wave traveling from left to right (the positive x-direction) on the string, and consider a particular point \( a \) on the string (Fig. 15.12a). The string to the left of \( a \) exerts a force on the string to the right of it, and vice versa. In Fig. 15.12b the string to the left of \( a \) has been removed, and the force it exerts at \( a \) is represented
by the components $F$ and $F_y$, as we did in Figs. 15.8 and 15.10. We note again that $F_y/F$ is equal to the negative of the slope of the string at $a$, which is also given by $\partial y/\partial x$. Putting these together, we have

$$F_y(x, t) = -F \frac{\partial y(x, t)}{\partial x}$$  \hspace{1cm} (15.20)

We need the negative sign because $F_y$ is negative when the slope is positive. We write the vertical force as $F_y(x, t)$ as a reminder that its value may be different at different points along the string and at different times.

When point $a$ moves in the $y$-direction, the force $F_y$ does work on this point and therefore transfers energy into the part of the string to the right of $a$. The corresponding power $P$ (rate of doing work) at the point $a$ is the transverse force $F_y(x, t)$ at $a$ times the transverse velocity $v_y(x, t) = \partial y(x, t)/\partial t$ of that point:

$$P(x, t) = F_y(x, t)v_y(x, t) = -F \frac{\partial y(x, t)}{\partial x} \frac{\partial y(x, t)}{\partial t}$$  \hspace{1cm} (15.21)

This power is the instantaneous rate at which energy is transferred along the string. Its value depends on the position $x$ on the string and on the time $t$. Note that energy is being transferred only at points where the string has a nonzero slope ($\partial y/\partial x$ is nonzero), so that there is a transverse component of the tension force, and where the string has a nonzero transverse velocity ($\partial y/\partial t$ is nonzero) so that the transverse force can do work.

Equation (15.21) is valid for any wave on a string, sinusoidal or not. For a sinusoidal wave with wave function given by Eq. (15.7), we have

$$y(x, t) = A \cos(kx - \omega t)$$

$$\frac{\partial y(x, t)}{\partial x} = -kA \sin(kx - \omega t)$$

$$\frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

$$P(x, t) = Fk\omega A^2 \sin^2(kx - \omega t)$$  \hspace{1cm} (15.22)

By using the relations $\omega = vk$ and $v^2 = F/\mu$, we can also express Eq. (15.22) in the alternative form

$$P(x, t) = \sqrt{\mu F \omega^2 A^2} \sin^2(kx - \omega t)$$  \hspace{1cm} (15.23)

The $\sin^2$ function is never negative, so the instantaneous power in a sinusoidal wave is either positive (so that energy flows in the positive $x$-direction) or zero (at points where there is no energy transfer). Energy is never transferred in the direction opposite to the direction of wave propagation (Fig. 15.13).

The maximum value of the instantaneous power $P(x, t)$ occurs when the $\sin^2$ function has the value unity:

$$P_{\text{max}} = \sqrt{\mu F \omega^2 A^2}$$  \hspace{1cm} (15.24)

To obtain the average power from Eq. (15.23), we note that the average value of the $\sin^2$ function, averaged over any whole number of cycles, is $1/2$. Hence the average power is

$$P_{\text{av}} = \frac{1}{2} \sqrt{\mu F \omega^2 A^2} \quad \text{(average power, sinusoidal wave on a string)}$$  \hspace{1cm} (15.25)

![Wave power versus time](image)
The average power is just one-half of the maximum instantaneous power (see Fig. 15.13).

The average rate of energy transfer is proportional to the square of the amplitude and to the square of the frequency. This proportionality is a general result for mechanical waves of all types, including seismic waves (see the photo that opens this chapter). For a mechanical wave, the rate of energy transfer quadruples if the frequency is doubled (for the same amplitude) or if the amplitude is doubled (for the same frequency).

Electromagnetic waves turn out to be a bit different. While the average rate of energy transfer in an electromagnetic wave is proportional to the square of the amplitude, just as for mechanical waves, it is independent of the value of $\omega$.

---

**Example 15.5**

**Power in a wave**

a) In Examples 15.2 and 15.3, at what maximum rate does Throcky put energy into the rope? That is, what is his maximum instantaneous power? b) What is his average power? c) As Throcky tires, the amplitude decreases. What is the average power when the amplitude has dropped to 7.50 mm?

**SOLUTION**

**IDENTIFY:** Our target variable in (a) is the maximum instantaneous power, while the target variable in (b) and (c) is the average power. As we've seen, these two quantities have different values for a sinusoidal wave. We'll be able to calculate the values of both quantities because we know all the other properties of the wave from Example 15.2.

**SET UP:** For part (a) we use Eq. (15.24), while for parts (b) and (c) we use Eq. (15.25).

**EXECUTE:** a) The maximum instantaneous power is

$$P_{\text{max}} = \sqrt{\mu F o^2 A^2}$$

$$= \sqrt{(0.250 \text{ kg/m}) (36.0 \text{ N}) (4.00\pi \text{ rad/s})^2 (0.075 \text{ m})^2}$$

$$= 2.66 \text{ W}$$

b) From Eqs. (15.24) and (15.25), the average power is one-half of the maximum instantaneous power, so

$$P_{\text{av}} = \frac{1}{2} (2.66 \text{ W}) = 1.33 \text{ W}$$

c) The new amplitude is $\frac{1}{10}$ of the value we used in parts (a) and (b). The average power is proportional to the square of the amplitude, so now the average power is

$$P_{\text{av}} = \left(\frac{1}{10}\right)^2 (1.33 \text{ W}) = 0.0133 \text{ W} = 13.3 \text{ mW}$$

**EVALUATE:** The maximum instantaneous power in part (a) occurs when the quantity $\sin^2(kx - o t)$ in Eq. (15.23) is equal to 1. At any given value of $x$, this happens twice per period of the wave—once when the sine function is equal to +1, once when it's equal to −1. The minimum instantaneous power is zero; this occurs when $\sin(kx - o t) = 0$, which also happens twice per period.

---

**Wave Intensity**

Waves on a string carry energy in just one dimension of space (along the direction of the string). But other types of waves, including sound waves in air and seismic waves in the body of the earth, carry energy across all three dimensions of space. For waves that travel in three dimensions, we define the intensity (denoted by $I$) to be the time average rate at which energy is transported by the wave, per unit area, across a surface perpendicular to the direction of propagation. That is, intensity $I$ is average power per unit area. It is usually measured in watts per square meter (W/m$^2$).

If waves spread out equally in all directions from a source, the intensity at a distance $r$ from the source is inversely proportional to $r^2$. (Fig. 15.14). This follows directly from energy conservation. If the power output of the source is...
then the average intensity $I_1$ through a sphere with radius $r_1$ and surface area $4\pi r_1^2$ is

$$I_1 = \frac{P}{4\pi r_1^2}$$

The average intensity $I_2$ through a sphere with a different radius $r_2$ is given by a similar expression. If no energy is absorbed between the two spheres, the power $P$ must be the same for both, and

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \quad \text{(inverse-square law for intensity)} \quad (15.26)$$

The intensity $I$ at any distance $r$ is therefore inversely proportional to $r^2$. This relationship is called the inverse-square law for intensity.

### Example 15.6

The inverse-square law

A tornado warning siren on top of a tall pole radiates sound waves uniformly in all directions. At a distance of 15.0 m the intensity of the sound is 0.250 W/m². At what distance from the siren is the intensity 0.010 W/m²?

**EXECUTE:** We solve Eq. (15.26) for $r_2$:

$$r_2 = r_1 \sqrt{\frac{I_1}{I_2}} = (15.0 \text{ m}) \sqrt{\frac{0.250 \text{ W/m}^2}{0.010 \text{ W/m}^2}} = 75.0 \text{ m}$$

**EVALUATE:** As a check on our answer, note that $r_2$ is five times greater than $r_1$. By the inverse-square law, the intensity $I_2$ should be $1/5^2 = 1/25$ as great as $I_1$, and indeed it is.

By using the inverse-square law we've assumed that the sound waves travel in straight lines away from the siren. A more realistic solution of this problem would account for the reflection of sound waves from the ground. Such a solution is beyond our scope, however.

### Test Your Understanding

In Example 15.5, suppose the wave amplitude remained equal to 0.075 m but the frequency increased from 2.00 Hz to 4.00 Hz. What effect would this have on the average power?

### 15.6 Wave Interference, Boundary Conditions, and Superposition

Up to this point we've been discussing waves that propagate continuously in the same direction. But when a wave strikes the boundaries of its medium, all or part of the wave is reflected. When you yell at a building wall or a cliff face some
distance away, the sound wave is reflected from the rigid surface and you hear an
echo. When you flip the end of a rope whose far end is tied to a rigid support, a
pulse travels the length of the rope and is reflected back to you. In both cases, the
initial and reflected waves overlap in the same region of the medium. This over-
lapping of waves is called **interference**. (In general, the term *interference* refers
to what happens when two or more waves pass through the same region at the
same time.)

As a simple example of wave reflections and the role of the boundary of a
wave medium, let's look again at transverse waves on a stretched string. What
happens when a wave pulse or a sinusoidal wave arrives at the *end* of the string?

If the end is fastened to a rigid support, it is a *fixed* end that cannot move. The
arriving wave exerts a force on the support; the reaction to this force, exerted by
the support on the string, “kicks back” on the string and sets up a *reflected* pulse
or wave traveling in the reverse direction. Figure 15.15 is a series of photographs
showing the reflection of a pulse at the fixed end of a long coiled spring. The
reflected pulse moves in the opposite direction from the initial, or *incident*, pulse,
and its displacement is also opposite. This situation is illustrated for a wave pulse
on a string in Fig. 15.16a.

**15.15** A series of images of a wave pulse,
equally spaced in time from top to bottom.
The pulse starts at the right in the top
image, travels to the left, and is reflected
from the fixed end at the left.

**15.16** Reflection of a wave pulse (a) at a fixed end of a string and (b) at a free end. Time
increases from top to bottom in each figure.
The opposite situation from an end that is held stationary is a *free* end, one that is perfectly free to move in the direction perpendicular to the length of the string. For example, the string might be tied to a light ring that slides on a frictionless rod perpendicular to the rope, as in Fig. 15.16b. The ring and rod maintain the tension but exert no transverse force. When a wave arrives at this free end, the ring slides along the rod. The ring reaches a maximum displacement, and both it and the string come momentarily to rest, as in drawing (4) in Fig. 15.16b. But the string is now stretched, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced (drawing (7)). As for a fixed end, the reflected pulse moves in the opposite direction from the initial pulse, but now the direction of the displacement is the same as for the initial pulse. The conditions at the end of the string, such as a rigid support or the complete absence of transverse force, are called *boundary conditions*.

The formation of the reflected pulse is similar to the overlap of two pulses traveling in opposite directions. Figure 15.17 shows two pulses with the same shape, one inverted with respect to the other, traveling in opposite directions. As the pulses overlap and pass each other, the total displacement of the string is the *algebraic sum* of the displacements at that point in the individual pulses. Because these two pulses have the same shape, the total displacement at point \( O \) in the middle of the figure is zero at all times. Thus the motion of the right half of the string would be the same if we cut the string at point \( O \), threw away the left side, and held the end at \( O \) fixed. The two pulses on the right side then correspond to the incident and reflected pulses, combining so that the total displacement at \( O \) is always zero. For this to occur, the reflected pulse must be inverted relative to the incident pulse.

Figure 15.18 shows two pulses with the same shape, traveling in opposite directions but not inverted relative to each other. The displacement at point \( O \) in the middle of the figure is not zero, but the slope of the string at this point is always zero. According to Eq. (15.20), this corresponds to the absence of any transverse force at this point. In this case the motion of the right half of the string would be the same as if we cut the string at point \( O \) and anchored the end with a frictionless sliding ring (Fig. 15.16b) that maintains tension without exerting any transverse force. In other words, this situation corresponds to reflection of a pulse at a free end of a string at point \( O \). In this case the reflected pulse is *not* inverted.

**The Principle of Superposition**

Combining the displacements of the separate pulses at each point to obtain the actual displacement is an example of the *principle of superposition*: when two waves overlap, the actual displacement of any point on the string at any time is obtained by adding the displacement the point would have if only the first wave were present and the displacement it would have if only the second wave were present. In other words, the wave function \( y(x, t) \) that describes the resulting motion in this situation is obtained by *adding* the two wave functions for the two separate waves:

\[
y(x, t) = y_1(x, t) + y_2(x, t) \quad \text{(principle of superposition)} \quad (15.27)
\]

Mathematically, this additive property of wave functions follows from the form of the wave equation, Eq. (15.12) or (15.18), which every physically possible wave function must satisfy. Specifically, the wave equation is *linear*: that is, it contains the function \( y(x, t) \) only to the first power (there are no terms involving
Two wave pulses with different shapes. As a result, if any two functions $y_1(x,t)$ and $y_2(x,t)$ satisfy the wave equation separately, their sum $y_1(x,t) + y_2(x,t)$ also satisfies it and is therefore a physically possible motion. Because this principle depends on the linearity of the wave equation and the corresponding linear-combination property of its solutions, it is also called the principle of linear superposition. For some physical systems, such as a medium that does not obey Hooke’s law, the wave equation is not linear; this principle does not hold for such systems.

The principle of superposition is of central importance in all types of waves. When a friend talks to you while you are listening to music, you can distinguish the sound of speech and the sound of music from each other. This is precisely because the total sound wave reaching your ears is the algebraic sum of the wave produced by your friend’s voice and the wave produced by the speakers of your stereo. If two sound waves did not combine in this simple linear way, the sound you would hear in this situation would be a hopeless jumble. Superposition also applies to electromagnetic waves (such as light) and many other types of waves.

Two wave pulses with different shapes are traveling in opposite directions along a string (Fig. 15.19). Make a series of sketches like Fig. 15.18 showing the shape of the string as the two pulses approach, overlap, and then pass each other.

15.7 | Standing Waves on a String

We have talked about the reflection of a wave pulse on a string when it arrives at a boundary point (either a fixed end or a free end). Now let’s look at what happens when a sinusoidal wave is reflected by a fixed end of a string. We’ll again approach the problem by considering the superposition of two waves propagating through the string, one representing the original or incident wave and the other representing the wave reflected at the fixed end.

Figure 15.20 shows a string that is fixed at its left-hand end. Its right-hand end is moved up and down in simple harmonic motion to produce a wave that travels to the left; the wave reflected from the fixed end travels to the right. The resulting motion when the two waves combine no longer looks like two waves traveling in opposite directions. The string appears to be subdivided into a number of segments, as in the time-exposure photographs of Figs. 15.20a, 15.20b, 15.20c, and 15.20d. Figure 15.20e shows two instantaneous shapes of the string in Fig. 15.20b. Let’s compare this behavior with the waves we studied in Sections 15.1 through 15.5. In a wave that travels along the string, the amplitude is constant and the wave pattern moves with a speed equal to the wave speed. Here, instead, the wave pattern remains in the same position along the string and its amplitude fluctuates. There are particular points called nodes (labeled $N$ in Fig. 15.20e) that never move at all. Midway between the nodes are points called antinodes (labeled $A$ in Fig. 15.20c) where the amplitude of motion is greatest. Because the wave pattern doesn’t appear to be moving in either direction along the string, it is called a standing wave. (To emphasize the difference, a wave that does move along the string is called a traveling wave.)

The principle of superposition explains how the incident and reflected wave combine to form a standing wave. In Fig. 15.21 the red curves show a wave traveling to the left. The blue curves show a wave traveling to the right with the same propagation speed, wavelength, and amplitude. The waves are shown at nine
(a)–(d) Time exposures of standing waves in a stretched string. From (a) to (d), the frequency of oscillation of the right-hand end increases and the wavelength of the standing wave decreases. (e) The extremes of the motion of the standing wave in (b), with nodes at the center and at the ends. The right-hand end of the string moves very little compared to the antinodes and so is essentially a node.

At certain instants, such as \( t = 4T/16 \), the two wave patterns are exactly in phase with each other, and the shape of the string is a sine curve with twice the amplitude of either individual wave. At other instants, such as \( t = 8T/16 \), the two waves are exactly out of phase with each other, and the total wave at that instant is zero. The resultant displacement is always zero at those places marked \( N \) at the bottom of Fig. 15.21. These are the nodes. At a node the displacements of the two waves in red and blue are always equal and opposite and cancel each other out. This cancellation is called destructive interference. Midway between the nodes are the points of greatest amplitude or antinodes, marked \( A \). At the antinodes the displacements of the two waves in red and blue are always identical, giving a large resultant displacement; this phenomenon is called constructive interference. We can see from the figure that the distance between successive nodes or between successive antinodes is one half-wavelength, or \( \lambda/2 \).

We can derive a wave function for the standing wave of Fig. 15.21 by adding the wave functions \( y_1(x, t) \) and \( y_2(x, t) \) for two waves with equal amplitude, period, and wavelength traveling in opposite directions. Here \( y_1(x, t) \) (the red curves in Fig. 15.21) represents an incoming, or incident, wave traveling to the left along the \( +x \)-axis, arriving at the point \( x = 0 \) and being reflected; \( y_2(x, t) \) (the blue curves in Fig. 15.21) represents the reflected wave traveling to the right from \( x = 0 \). We noted in Section 15.6 that the wave reflected from a fixed end of a string is inverted, so we give a negative sign to one of the waves:

\[
\begin{align*}
y_1(x, t) &= -A \cos(kx + wt) \quad \text{(incident wave traveling to the left)} \\
y_2(x, t) &= A \cos(kx - wt) \quad \text{(reflected wave traveling to the right)}
\end{align*}
\]
Formation of a standing wave. A wave traveling to the left (red curves) combines with a wave traveling to the right (blue curves) to form a standing wave (black curves). The horizontal x-axis in each part shows the equilibrium position of the string.

Note also that the change in sign corresponds to a shift in phase of 180° or \( \pi \) radians. At \( x = 0 \) the motion from the reflected wave is \( A \cos \omega t \); and the motion from the incident wave is \( -A \cos \omega t \), which we can also write as \( A \cos(\omega t + \pi) \). From Eq. (15.27), the wave function for the standing wave is the sum of the individual wave functions:

\[
y(x, t) = y_1(x, t) + y_2(x, t) = A \left[ -\cos(kx + \omega t) + \cos(kx - \omega t) \right]
\]
Standing Waves on a String

We can rewrite each of the cosine terms by using the identities for the cosine of the sum and difference of two angles: \( \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \). Applying these and combining terms, we obtain the wave function for the standing wave:

\[
y(x, t) = y_1(x, t) + y_2(x, t) = (2A \sin kx) \sin \omega t
\]

or

\[
y(x, t) = (A_{SW} \sin kx) \sin \omega t
\]

(standing wave on a string, fixed end at \( x = 0 \))

The standing wave amplitude \( A_{SW} \) is twice the amplitude \( A \) of either of the original traveling waves:

\[
A_{SW} = 2A
\]

Equation (15.28) has two factors: a function of \( x \) and a function of \( t \). The factor \( A_{SW} \) shows that at each instant the shape of the string is a sine curve. But unlike a wave traveling along a string, the wave shape stays in the same position, oscillating up and down as described by the \( \sin \omega t \) factor. This behavior is shown graphically by the black curves in Fig. 15.21. Each point in the string still undergoes simple harmonic motion, but all the points between any successive pair of nodes oscillate in phase. This is in contrast to the phase differences between oscillations of adjacent points that we see with a wave traveling in one direction.

We can use Eq. (15.28) to find the positions of the nodes; these are the points for which \( \sin kx = 0 \), so the displacement is always zero. This occurs when \( kx = 0, \pi, 2\pi, 3\pi, \ldots \), or, using \( k = \frac{2\pi}{\lambda} \),

\[
x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots
\]

\[
= 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots
\]

(nodes of a standing wave on a string, fixed end at \( x = 0 \))

In particular, there is a node at \( x = 0 \), as there should be, since this point is a fixed end of the string.

A standing wave, unlike a traveling wave, does not transfer energy from one end to the other. The two waves that form it would individually carry equal amounts of power in opposite directions. There is a local flow of energy from each node to the adjacent antinodes and back, but the average rate of energy transfer is zero at every point. If you evaluate the wave power given by Eq. (15.21) using the wave function of Eq. (15.28), you will find that the average power is zero (see Challenge Problem 15.82).

**Standing Waves**

**Problem-Solving Strategy**

**IDENTIFY the relevant concepts:** As with traveling waves, it’s useful to distinguish between the purely kinematic quantities, such as wave speed \( v \), wavelength \( \lambda \), and frequency \( f \), and the dynamic quantities involving the properties of the medium, such as \( F \) and \( \mu \) for transverse waves on a string. Once you decide what the target variable is, try to determine whether the problem is only kinematic in nature or whether the properties of the medium are also involved.
**SET UP** the problem using the following steps:

1. In visualizing nodes and antinodes in standing waves, it is always helpful to draw diagrams. For a string you can draw the shape at one instant and label the nodes $N$ and antinodes $A$. The distance between two adjacent nodes or two adjacent antinodes is always $\lambda/2$, and the distance between a node and the adjacent antinode is always $\lambda/4$.

2. Decide which equations you’ll need to use. The wave function for the standing wave is almost always useful (like Eq. (15.28)).

3. You can compute the wave speed if you know either $\lambda$ and $f$ (or, equivalently, $k = 2\pi/\lambda$ and $\omega = 2\pi f$) or the properties of the medium (for a string, $F$ and $\mu$).

**EXECUTE** the solution as follows:

Solve for the unknown quantities using the equations you’ve selected. Once you have the wave function, you can find the value of the displacement $y$ at any point in the wave medium (value of $x$) and at any time. You can find the velocity of a particle in the wave medium by taking the partial derivative of $y$ with respect to time. To find the acceleration of such a particle, take the second partial derivative of $y$ with respect to time.

**EVALUATE** your answer: Compare your numerical answers with your diagram. Check that the wave function is compatible with the boundary conditions (for example, the displacement should be zero at a fixed end).

---

**Example 15.7 Standing waves on a guitar string**

One of the strings of a guitar lies along the x-axis when in equilibrium. The end of the string at $x = 0$ (the bridge of the guitar) is tied down. An incident sinusoidal wave, corresponding to the red curves in Fig. 15.21, travels along the string in the $-x$-direction at 143 m/s with an amplitude of 0.750 mm and a frequency of 440 Hz. This wave is reflected from the fixed end at $x = 0$, and the superposition of the incident traveling wave and the reflected traveling wave forms a standing wave. a) Find the equation giving the displacement of a point on the string as a function of position and time. b) Locate the points on the string that don’t move at all. c) Find the amplitude, maximum transverse velocity, and maximum transverse acceleration at points of maximum oscillation.

**SOLUTION**

**IDENTIFY:** This is a kinematics problem in which we are asked to describe the motion of the string (see the Problem-Solving Strategy in Section 15.3). The target variables are the wave function of the standing wave (part a), the locations of the points that don’t move, or nodes (part b), and the maximum values of displacement, maximum transverse velocity $v$, and transverse acceleration $a$ (Waves on a string are transverse waves, so transverse means "in the direction of the displacement," that is, in the $y$-direction.) To find these quantities we use the expression that we derived in this section for a standing wave on a string with a fixed end, as well as other relations from Sections 15.2 and 15.3.

**SET UP:** Since there is a fixed end at $x = 0$, we may use Eqs. (15.28) and (15.29) to describe this standing wave. We also use the relationships among $\omega$, $k$, $f$, $\lambda$, and the wave speed $v$.

**EXECUTE:** a) To use Eq. (15.28) we need the values of $A_{kw}$, $\omega$, and $k$. The amplitude of the incident wave is $A = 0.750$ mm $= 7.50 \times 10^{-4}$ m; the reflected wave has the same amplitude, and the standing wave amplitude is $A_{kw} = 2A = 1.50 \times 10^{-3}$ m. The angular frequency $\omega$ and wave number $k$ are

$$\omega = 2\pi f = \left(\frac{2\pi}{\lambda}\right) \left(440 \text{ s}^{-1}\right) = 2760 \text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{2760 \text{ rad/s}}{143 \text{ m/s}} = 19.3 \text{ rad/m}$$

Then Eq. (15.28) gives

$$y(x, t) = \left[A_{kw} \sin(kx)\right] \sin(\omega t)$$

$$= \left[(1.50 \times 10^{-3}) \sin(19.3 \text{ rad/m} x)\right] \sin(2760 \text{ rad/s} t)$$

b) The positions of the nodes are given by Eq. (15.29):

$$x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$$

The wavelength is

$$\lambda = \frac{v}{f} = \frac{143 \text{ m/s}}{440 \text{ Hz}} = 0.325 \text{ m}$$

so the nodes are at the following distances from the fixed end:

$$x = 0, 0.163 \text{ m}, 0.325 \text{ m}, 0.488 \text{ m}, \ldots$$

c) From the expression in part (a) for $y(x, t)$, we see that the maximum displacement from equilibrium is $1.50 \times 10^{-3} m = 1.50$ mm, which is just twice the amplitude of the incident wave. This maximum occurs at the antinodes, which are midway between adjacent nodes (that is, at $x = 0.081 \text{ m}, 0.244 \text{ m}, 0.406 \text{ m}, \ldots$).

For a particle on the string at any point $x$, the transverse ($y$-) velocity is

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t}$$

$$= \left[(1.50 \times 10^{-3}) \sin(19.3 \text{ rad/m} x)\right] \left[2760 \text{ rad/s} \cos(2760 \text{ rad/s} t)\right]$$

$$= \left[(4.15 \text{ m/s}) \sin(19.3 \text{ rad/m} x)\right] \cos(2760 \text{ rad/s} t)$$

At an antinode, $\sin(19.3 \text{ rad/m} x) = \pm 1$ and the transverse velocity varies in value between 4.15 m/s and $-4.15$ m/s. As is always the case in simple harmonic motion, the maximum velocity occurs when the particle is passing through the equilibrium position $(y = 0)$. 

---
The transverse acceleration $a_y(x, t)$ is the first partial derivative of $y(x, t)$ with respect to time (that is, the second partial derivative of $y(x, t)$ with respect to time). We leave the calculation to you; the result is

$$a_y(x, t) = \frac{\partial v_y(x, t)}{\partial t} = \frac{\partial^2 y(x, t)}{\partial t^2}$$

$= (\frac{1.15 \times 10^4 \text{ m/s}^2}{2}) \sin (19.3 \text{ rad/m})x$

$\times \sin (2760 \text{ rad/s})t$

At the antinodes, the transverse acceleration varies in value between $+1.15 \times 10^4 \text{ m/s}^2$ and $-1.15 \times 10^4 \text{ m/s}^2$.

**EVALUATE:** The maximum transverse velocity at an antinode is quite respectable (about 15 km/h, or 9.3 mi/h). But the maximum transverse acceleration is tremendous, 1170 times the acceleration due to gravity! Guitar strings are made of sturdy stuff to be able to withstand such acceleration.

Guitar strings are actually tied down at both ends. We’ll see the consequences of this in the next section.

---

**Test Your Understanding**

Suppose the frequency of the standing wave in Example 15.7 were doubled from 440 Hz to 880 Hz. Would all of the nodes for $f = 440 \text{ Hz}$ also be nodes for $f = 880 \text{ Hz}$? If so, would there be additional nodes for $f = 880 \text{ Hz}$? If not, which nodes are absent for $f = 880 \text{ Hz}$?

---

15.8 | **Normal Modes of a String**

When we described standing waves on a string rigidly held at one end, as in Fig. 15.20, we made no assumptions about the length of the string or about what was happening at the other end. Let’s now consider a string of a definite length $L$, rigidly held at both ends. Such strings are found in many musical instruments, including pianos, violins, and guitars. When a guitar string is plucked, a wave is produced in the string; this wave is reflected and re-reflected from the ends of the string, making a standing wave. This standing wave on the string in turn produces a sound wave in the air, with a frequency determined by the properties of the string. This is what makes stringed instruments so useful in making music.

To understand these properties of standing waves on a string fixed at both ends, let’s first examine what happens when we set up a sinusoidal wave on such a string. The standing wave that results must have a node at both ends of the string. We saw in the preceding section that adjacent nodes are one half-wavelength ($\lambda/2$) apart, so the length of the string must be $\lambda/2$, or $2(\lambda/2)$, or $3(\lambda/2)$, or in general some integer number of half-wavelengths:

$$L = \frac{n \lambda}{2} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \quad (15.30)$$

That is, if a string with length $L$ is fixed at both ends, a standing wave can exist only if its wavelength satisfies Eq. (15.30).

Solving this equation for $\lambda$ and labeling the possible values of $\lambda$ as $\lambda_n$, we find

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \quad (15.31)$$

Waves can exist on the string if the wavelength is not equal to one of these values, but there cannot be a steady wave pattern with nodes and antinodes, and the total wave cannot be a standing wave. Equation (15.31) is illustrated by the standing waves shown in Figs. 15.20a, 15.20b, 15.20c, and 15.20d; these represent $n = 1, 2, 3, \text{ and } 4$, respectively.
Corresponding to the series of possible standing-wave wavelengths $\lambda_n$ is a series of possible standing-wave frequencies $f_n$, each related to its corresponding wavelength by $f_n = \frac{v}{\lambda_n}$. The smallest frequency $f_1$ corresponds to the largest wavelength (the $n = 1$ case), $\lambda_1 = 2L$:

$$f_1 = \frac{v}{2L} \quad (\text{string fixed at both ends}) \quad (15.32)$$

This is called the fundamental frequency. The other standing-wave frequencies are $f_2 = \frac{2v}{2L}$, $f_3 = \frac{3v}{2L}$, and so on. These are all integer multiples of the fundamental frequency $f_1$, such as $2f_1$, $3f_1$, $4f_1$, and so on, and we can express all the frequencies as

$$f_n = n \frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots) \quad (\text{string fixed at both ends}) \quad (15.33)$$

These frequencies are called harmonics, and the series is called a harmonic series. Musicians sometimes call $f_2$, $f_3$, and so on overtones; $f_2$ is the second harmonic or the first overtone, $f_3$ is the third harmonic or the second overtone, and so on. The first harmonic is the same as the fundamental frequency (Fig. 15.22).

For a string with fixed ends at $x = 0$ and $x = L$, the wave function $y(x, t)$ of the $n$th standing wave is given by Eq. (15.28) (which satisfies the condition that there is a node at $x = 0$), with $\omega = \omega_n = 2\pi f_n$ and $k = k_n = 2\pi/\lambda_n$:

$$y_n(x, t) = A_{SW} \sin k_n x \sin \omega_n t \quad (15.34)$$

You can easily show that this wave function has nodes at both $x = 0$ and $x = L$, as it must.

A normal mode of an oscillating system is a motion in which all particles of the system move sinusoidally with the same frequency. For a system made up of a string of length $L$ fixed at both ends, each of the wavelengths given by Eq. (15.31) corresponds to a possible normal-mode pattern and frequency. There are infinitely many normal modes, each with its characteristic frequency and vibration pattern. Figure 15.23 shows the first four normal-mode patterns and their associated frequencies and wave-
lengths; these correspond to Eq. (15.34) with \( n = 1, 2, 3, \) and \( 4 \). By contrast, a harmonic oscillator, which has only one oscillating particle, has only one normal mode and one characteristic frequency. The string fixed at both ends has infinitely many normal modes because it is made up of a very large (effectively infinite) number of particles. More complicated oscillating systems also have infinite numbers of normal modes, though with more complex normal-mode patterns than a string (Fig. 15.24).

If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode. Such a vibrating string would displace the surrounding air with the same frequency, producing a traveling sinusoidal sound wave that your ears would perceive as a pure tone. But when a string is struck (as in a piano) or plucked (as is done to guitar strings), the shape of the displaced string is not as simple as one of the patterns in Fig. 15.23. The fundamental as well as many overtones are present in the resulting vibration. This motion is therefore a combination or superposition of many normal modes. Several simple-harmonic motions of different frequencies are present simultaneously, and the displacement of any point on the string is the sum (or superposition) of displacements associated with the individual modes. The sound produced by the vibrating string is likewise a superposition of traveling sinusoidal sound waves, which you perceive as a rich, complex tone with the fundamental frequency \( f_1 \). The standing wave on the string and the traveling sound wave in the air have similar harmonic content (the extent to which frequencies higher than the fundamental are present). The harmonic content depends on how the string is initially set into motion. If you pluck the strings of an acoustic guitar in the normal location over the sound hole, the sound that you hear has a different harmonic content than if you pluck the strings next to the fixed end on the guitar body.

It is possible to represent every possible motion of the string as some superposition of normal-mode motions. Finding this representation for a given vibration pattern is called harmonic analysis. The sum of sinusoidal functions that represents a complex wave is called a Fourier series. Figure 15.25 shows how a standing wave that is produced by plucking a guitar string of length \( L \) at a point \( L/4 \) from one end can be represented as a combination of sinusoidal functions.

As we have seen, the fundamental frequency of a vibrating string is \( f_1 = \frac{v}{2L} \). The speed \( v \) of waves on the string is determined by Eq. (15.13), \( v = \sqrt{\frac{F}{\mu}} \). Combining these, we find

\[
f_1 = \frac{1}{2L} \sqrt{\frac{F}{\mu}} \quad \text{(string fixed at both ends)} \tag{15.35}
\]

10.10 Complex Waves: Fourier Analysis

Astronomers have discovered that the sun oscillates in several different normal modes. In this computer simulation of one such mode, blue denotes places where solar material is moving outward and red denotes where it is moving inward.

When a guitar string is plucked (pulled into a triangular shape) and released, a standing wave results. The standing wave is well represented (except at the sharp maximum point) by the sum of just three sinusoidal functions. Including additional sinusoidal functions further improves the representation.
Comparing the range of a concert grand piano to those of a violin (red bar), a viola (yellow bar), a cello (green bar), and a string bass (blue bar). In all cases, longer strings provide bass notes and shorter strings produce treble notes. This is also the fundamental frequency of the sound wave created in the surrounding air by the vibrating string. Familiar musical instruments show how \( f \), depends on the properties of the string. The inverse dependence of frequency on length \( L \) is illustrated by the long strings of the bass (low-frequency) section of the piano or the bass viol compared with the shorter strings on the treble section of the piano or the violin (Fig. 15.26). The pitch of a violin or guitar is usually varied by pressing a string against the fingerboard with the fingers to change the length \( L \) of the vibrating portion of the string. Increasing the tension \( F \) increases the wave speed \( u \) and thus increases the frequency (and the pitch). All string instruments are “tuned” to the correct frequencies by varying the tension; you tighten the string to raise the pitch. Finally, increasing the mass per unit length \( \mu \) decreases the wave speed and thus the frequency. The lower notes on a steel guitar are produced by thicker strings, and one reason for winding the bass strings of a piano with wire is to obtain the desired low frequency without resorting to a string that is inconveniently long.

Wind instruments such as saxophones and trombones also have normal modes. As for stringed instruments, the frequencies of these normal modes determine the pitch of the musical tones that these instruments produce. We’ll discuss these instruments and many other aspects of sound in Chapter 16.

Example 15.8
A giant bass viol

In an effort to get your name in the Guinness Book of World Records, you set out to build a bass viol with strings that have a length of 5.00 m between fixed points. One string has a linear mass density of 40.0 g/m and a fundamental frequency of 20.0 Hz (the lowest frequency that the human ear can hear). Calculate (a) the tension of this string, (b) the frequency and wavelength on the string of the second harmonic, and (c) the frequency and wavelength on the string of the second overtone.

SOLUTION

IDENTIFY: The target variable in (a) is the string tension; we find this from the expression for the fundamental frequency of the string, which involves the tension. In parts (b) and (c) the target variables are the frequency and wavelength of different harmonics. We determine these from the given length of the string and fundamental frequency.
SET UP: For part (a), the equation to use is Eq. (15.35); it involves the known values of $f_1$, $L$, and $\mu$ as well as the target variable $F$. We solve parts (b) and (c) using Eqs. (15.31) and (15.33).

EXECUTE: a) We solve Eq. (15.35) for the string tension $F$:

$$ F = 4\mu L f_1^2 = 4(40.0 \times 10^{-3} \text{ kg/m})(5.00 \text{ m})^2(20.0 \text{ s}^{-1})^2 $$

$$ = 1600 \text{ N} = 360 \text{ lb} $$

b) The second harmonic is denoted by $n = 2$. From Eq. (15.33), the second harmonic frequency is

$$ f_2 = 2f_1 = 2(20.0 \text{ Hz}) = 40.0 \text{ Hz} $$

From Eq. (15.31), the wavelength on the string of the second harmonic is

$$ \lambda_2 = \frac{2L}{2} = 5.00 \text{ m} $$

c) The second overtone is the “second tone over” (above) the fundamental, that is, $n = 3$. Its frequency and wavelength are

$$ f_3 = 3f_1 = 3(20.0 \text{ Hz}) = 60.0 \text{ Hz} $$

$$ \lambda_3 = \frac{2L}{3} = 3.33 \text{ m} $$

EVALUATE: The tension in part (a) is a bit larger than in a real bass viol, for which the string tension is typically a few hundred newtons. The wavelengths in parts (b) and (c) are equal to the length of the string and two-thirds the length of the string respectively; these results agree with the drawings of standing waves in Fig. 15.23.

Example 15.9

From waves on a string to sound waves in air

What are the frequency and wavelength of the sound waves produced in the air when the string in the previous example is vibrating at its fundamental frequency? The speed of sound in air at 20°C is 344 m/s.

SOLUTION

IDENTIFY: Our target variables are $f$ and $\lambda$ for the sound wave produced by the bass viol, not for the standing wave on the string. However, when the string vibrates at a particular frequency, the surrounding air is forced to vibrate at the same frequency. So the frequency of the sound wave is the same as that of the standing wave on the string. However, the relationship $\lambda = \frac{\nu}{f}$ shows that the wavelength of the sound wave is typically different from the wavelength of the standing wave on the string, because the two waves have different speeds.

SET UP: The only equation we need is $\nu = \lambda f$. We apply this to both the standing wave on the string (speed $\nu_{\text{string}}$) and the traveling sound wave (speed $\nu_{\text{sound}}$).

EXECUTE: The sound wave frequency is the same as the standing wave fundamental frequency: $f = f_1 = 20.0 \text{ Hz}$. The wavelength of the sound wave is

$$ \lambda_{\text{(sound)}} = \frac{\nu_{\text{sound}}}{f_1} = \frac{344 \text{ m/s}}{20.0 \text{ Hz}} = 17.2 \text{ m} $$

EVALUATE: Note that $\lambda_{\text{(sound)}}$ is greater than the wavelength of the standing wave on the string, $\lambda_{\text{(string)}} = 2L = 2(5.00 \text{ m}) = 10.0 \text{ m}$. Fundamentally this is because the speed of sound is greater than the speed of waves on the string. $\nu_{\text{string}} = \lambda_{\text{(string)}} f_1 = (10.0 \text{ m})(20.0 \text{ Hz}) = 200 \text{ m/s}$. Hence, for any normal mode on this string, the sound wave that is produced has the same frequency as the wave on the string but a wavelength that is greater by a factor of $\nu_{\text{sound}}/\nu_{\text{string}} = (344 \text{ m/s})/(200 \text{ m/s}) = 1.72$.

Test Your Understanding

While a guitar string is vibrating, you gently touch the midpoint of the string to ensure that the string does not vibrate at that point. Which normal modes cannot be present on the string while you are touching it in this way?
A wave is any disturbance from an equilibrium condition that propagates from one region to another. A mechanical wave always travels within some material called the medium. The wave disturbance propagates at the wave speed \( v \), which depends on the type of wave and the properties of the medium.

In a periodic wave, the motion of each point of the medium is periodic. A sinusoidal wave is a special periodic wave in which each point moves in simple harmonic motion. For any periodic wave, the frequency \( f \) is the number of cycles per unit time, the period \( T \) is the time for one cycle, the wavelength \( \lambda \) is the distance over which the wave pattern repeats, and the amplitude \( A \) is the maximum displacement of a particle in the medium. The product of \( \lambda \) and \( f \) equals the wave speed. (See Example 15.1)

The wave function \( y(x, t) \) describes the displacements of individual particles in the medium. Equations (15.3), (15.4), and (15.7) give the wave equation for a sinusoidal wave traveling in the \( +x \)-direction. If the wave is moving in the \( -x \)-direction, the minus signs in the cosine functions are replaced by plus signs. (See Example 15.2)

The wave function obeys a partial differential equation called the wave equation.

The speed of transverse waves on a string depends on the tension \( F \) and mass per unit length \( \mu \). (See Examples 15.3 and 15.4)

Wave motion conveys energy from one region to another. For a sinusoidal mechanical wave, the average power \( P_{av} \) is proportional to the square of the wave amplitude and the square of the frequency. For waves that spread out in three dimensions, the wave intensity \( I \) is inversely proportional to the distance from the source. (See Examples 15.5 and 15.6)
A wave that reaches a boundary of the medium in which it propagates is reflected. The principle of superposition states that the total wave displacement at any point where two or more waves overlap is the sum of the displacements of the individual waves.

When a sinusoidal wave is reflected from a fixed or free end of a stretched string, the incident and reflected waves combine to form a standing sinusoidal wave with nodes and antinodes. Adjacent nodes are spaced a distance $\lambda/2$ apart, as are adjacent antinodes. (See Example 15.7)

When both ends of a string with length $L$ are held fixed, standing waves can occur only when $L$ is an integer multiple of $\lambda/2$. Each frequency and its associated vibration pattern is called a normal mode. The lowest frequency $f_1$ is called the fundamental frequency. (See Examples 15.8 and 15.9)

Key Terms

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Your Notes