Integral

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{(-c)^{1/2}} \sin^{-1} \frac{bx+2c}{|x|(b^2-4ac)^{1/2}}$$

$a = 2\mu E$ $\quad b = 2\mu C$ $\quad -\ell^2$ when $c < 0$

$$-\ell^2 \text{ is } < 0$$

when $b^2 > 4ac$

$$4\mu^2 C^2 > 4(2\mu E)(-\ell^2)$$

$$\underbrace{-E < \frac{1}{2}\mu \frac{C^2}{\ell^2}}_{\text{only limiting when } E<0}$$

smallest E, largest $-E \rightarrow$ circular orbit

$$\rightarrow \frac{-\ell^2}{\mu R^3} + \frac{C}{R^2} = 0$$

$$\ell^2 = \mu R C$$

$$-E = \underbrace{+\frac{1}{2}\frac{C}{R}}_{\text{earlier}} \underbrace{\overset{?}{<} \frac{1}{2}\mu\frac{C^2}{\mu R C} \overset{?}{=} \frac{1}{2}\frac{C}{R}}_{\text{equal!}}$$

So, $b^2 = 4ac \rightarrow$ circular orbit.

$b^2 > 4ac$ when an eensy bit above.

$$\theta - \theta_0 = \sin^{-1}\left(\frac{\cancel{2}\nu c r - \cancel{2}\ell^2}{r\sqrt{(\cancel{2}\nu c)^2 - \cancel{4}(2\nu E)(\cancel{-}\ell^2)}}\right)$$

$$\sin(\theta - \theta_0) = \frac{\nu c - \ell^2/r}{\sqrt{(\nu c)^2 + 2\nu E \ell^2}}$$

$$\sin(\theta - \theta_0) = \frac{1 - \frac{\ell^2}{\nu c r}}{\sqrt{1 + \frac{2E\ell^2}{\nu c^2}}}$$

$$1 - \frac{\ell^2}{\nu c r} = \sqrt{1 + \frac{2E\ell^2}{\nu c^2}}\,\sin(\theta - \theta_0)$$

$$\Downarrow$$

$$r = \frac{\ell^2/\nu c}{1 - \left[1 + \frac{2E\ell^2}{\nu c^2}\right]^{1/2}\sin(\theta - \theta_0)}$$

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

(1) $\theta_0 = \frac{\pi}{2}$, $\sin(\theta - \theta_0) = \sin(\theta - \frac{\pi}{2})$

$$= \cos(-\theta) = \cos\theta$$

(2) $r_0 \equiv \frac{\ell^2}{\mu C}$ = radius of the circular orbit sharing that angular momentum ℓ

(3) $$\varepsilon = \sqrt{1 + \frac{E}{\left(\frac{\mu C^2}{2\ell^2}\right)}}$$

$$\frac{\mu C^2}{2\ell^2} \Rightarrow \underbrace{\frac{1}{2}\frac{G m_1 m_2}{r_0}}_{|-E_c|} = \frac{1}{2}\frac{C}{\frac{\ell^2}{\mu C}} = \frac{1}{2}\frac{\mu C^2}{\ell^2}$$

where E_c = total energy of circular orbit

so $$\varepsilon = \sqrt{1 + \frac{E}{|E_c|}}$$

Analysis: take ℓ as given...

• Compute $E = K + U(r)$

$$E = \underbrace{\frac{1}{2}\mu \dot{r}^2}_{} + \underbrace{\frac{\ell^2}{2\mu r^2} - \frac{C}{r}}_{}$$

Then $\dot{r}^2_{max}$ $\Leftarrow$ E_c for circular orbit, when $r = r_0$

$$\frac{E}{|E_c|} = \frac{\frac{1}{2}\mu \dot{r}^2_{max}}{|E_c|} + \underbrace{\frac{E_c}{|E_c|}}_{-1}$$

$$\varepsilon = \left| 1 + \frac{\frac{1}{2}\mu \dot{r}^2_{max}}{|E_c|} - 1 \right|$$

$$\varepsilon = \frac{\mu \dot{r}^2_{max}}{2|E_c|}$$

(i) Circular orbit.. $\dot{r}_{max} = 0$!

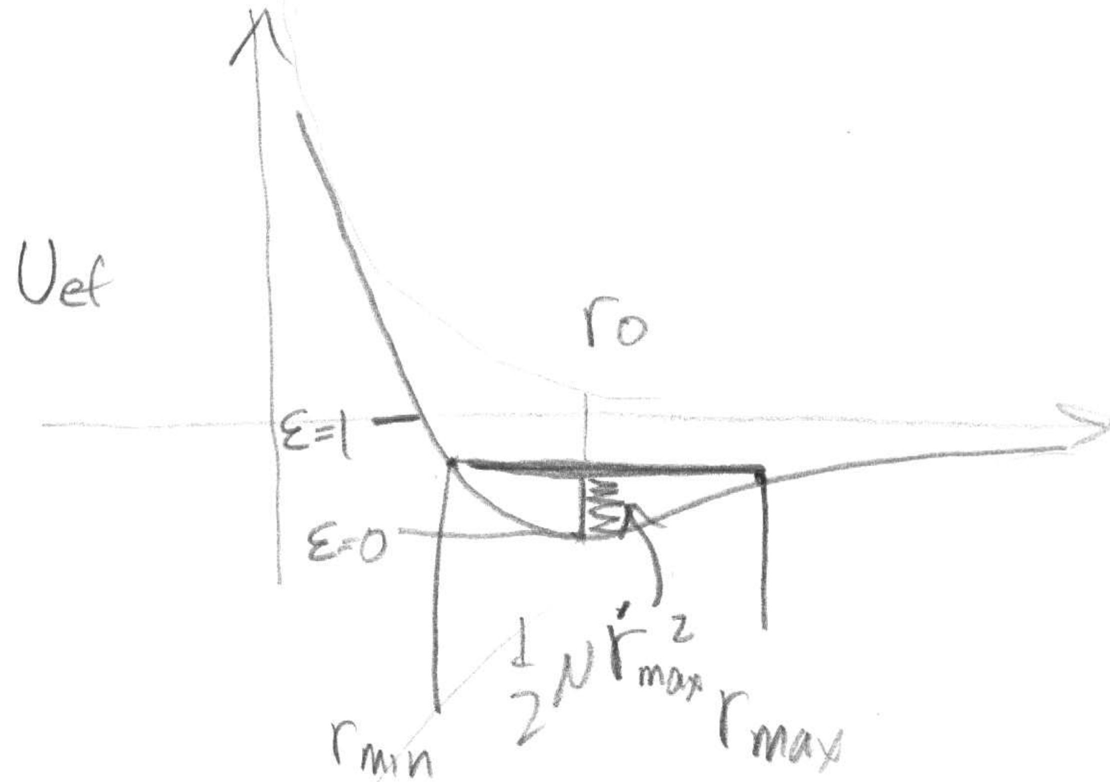
$$\varepsilon = 0$$

$$r(\theta) = \frac{r_0}{1 + 0 \cdot \sin(\theta - \theta_0)}$$

$$r(\theta) = r_0$$

(2) Elliptical Orbit

$$0 < \varepsilon < 1$$

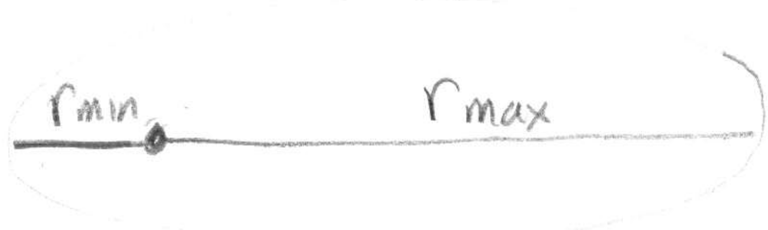


periapsis (perihelion, perigee)

apoapsis (aphelion, apogee)

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

$$r_{min} = \frac{r_0}{1+\varepsilon} \qquad r_{max} = \frac{r_0}{1-\varepsilon}$$



$$B = \frac{2r_0}{\sqrt{1-\varepsilon^2}}$$

$$A = \frac{r_0}{1+\varepsilon} + \frac{r_0}{1-\varepsilon}$$

$$\frac{r_0(1-\varepsilon) + r(1+\varepsilon)}{(1-\varepsilon^2)}$$

$$= \frac{2r_0}{1-\varepsilon^2}$$

not proved hear

(3) Parabolic Orbit..

$$\varepsilon = 1 \quad \rightarrow \quad \tfrac{1}{2}\mu\dot{r}^2 = |E_c|$$

$$E_{tot} = 0$$

$$r = \frac{r_0}{1-\cos\theta}$$